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ON PERIODIC WATER-WAVES AND THEIR CONVERGENCE TO SOLITARY WAVES IN THE LONG-WAVE LIMIT

By C. J. AMICK† and J. F. TOLAND‡
† *Department of Mathematics, University of Chicago, Chicago, Illinois* 60637 *U.S.A.*
‡ *Department of Mathematics, University College London, London WC1E 6BT*

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CONTENTS

A detailed discussion of Nekrasov's approach to the steady water-wave problems leads to a new integral equation formulation of the periodic problem. This development allows the adaptation of the methods of Amick & Toland (1981) to show the convergence of periodic waves to solitary waves in the long-wave limit.

In addition, it is shown how the classical integral equation formulation due to Nekrasov leads, via the Maximum Principle, to new results about qualitative features of periodic waves for which there has long been a global existence theory (Krasovskii 1961, Keady & Norbury 1978).

1. Introduction

1.1. *Introductory remarks*

Under consideration are the steady two-dimensional waves which can arise as the free surface of a heavy, ideal liquid acted on by gravity, and contained in a channel of infinite extent with a horizontal bottom, in the absence of surface tension effects. It is well known that both periodic waves (Keady & Norbury 1978; Krasovskii 1961) and solitary waves (Amick & Toland 1981; hereinafter referred to as I) of large amplitude may occur in these circumstances. A precise account of the free boundary-value problem presented by this situation is given in the next section, and various physical parameters describing the flow are introduced. After some basic results about conformal mappings and Jacobi elliptic functions have been recorded in § 1.3, the method of Nekrasov (1967) is used to reduce the existence question for these free boundary-value problems to a similar question for nonlinear integral equations. Throughout this section, we emphasize the role which various physical parameters play in these integral equation formulations. For example, in the periodic case, the *wavelength* and the *mean depth* (which is defined in § 1.2, and which is the quantity referred to as the undisturbed depth in Cokelet (1977)) are specified *a priori* and appear as constants in the equations, whereas other quantities such as *mean velocity*, the *flux* and the *flow velocity at the crest* depend on the solution of the equation being considered. An account of this is given in theorems 1.5 and 1.6.

Of the two integral equation formulations (1.31) and (1.32) of the periodic problems given in § 1.3, equation (1.32) is perhaps the more familiar. Keady *et al.* (1978) used it to prove a global existence theorem for periodic water-waves (though the physical interpretation of its solutions there is different from ours). Equation (1.31), which is equivalent to the usual integral equations for periodic wave (Krasovskii 1961; Nekrasov 1967; Milne-Thomson 1968), has distinct advantages for our purposes in § 3. The most important of these is its striking resemblance to the approximation used in I, § 3.2 to prove the existence of large-amplitude solitary waves.

After a few remarks in § 2.1 about recent developments in the theory of large-amplitude periodic water-waves, § 2.2 is devoted to a summary and sketch of the proofs of a global bifurcation theorem for periodic waves of wavelength λ on a flow of mean depth h, where λ and h are any given positive real numbers. Among these results is the existence of a connected set of such waves containing waves of all amplitudes up to that of a wave of extreme form. This connected set contains a wave whose maximum angle of inclination to the horizontal is β, for any $\beta \in [0, \frac{1}{6}\pi + \epsilon]$ where $\epsilon > 0$ is sufficiently small, and the mean velocity of all such waves is bounded away from zero and infinity. Some of these results are already known in a different context, while for others the proof given here is new. For the sake of clarity, we have collected them here and expressed them in terms of equation (1.31), which is the form in which we shall need them again in § 3.

In § 2.3 we show that solutions of equation (1.32) lie in a cone which is smaller than the cone of non-negative functions in $C_0[0, \frac{1}{2}\lambda]$, namely the cone $\hat{\mathscr{K}}$ of non-negative functions u which are decreasing on $[\frac{1}{4}\lambda, \frac{1}{2}\lambda]$ and such that $u(\chi) \geqslant u(\frac{1}{2}\lambda - \chi)$, $\chi \in [0, \frac{1}{4}\lambda]$. This leads to a considerable improvement in the global bifurcation theory for (1.32). We show that the maximal connected subset of non-trivial solutions which bifurcates from the curve of trivial solutions $\{(\mu, 0)\colon \mu \in \mathbb{R}\}$ at the first characteristic value, $6\pi\Lambda\lambda^{-1} \coth(2\pi h/\lambda)$, of the linearized problem is unbounded, and lies in $(6\pi\Lambda\lambda^{-1} \coth(2\pi h/\lambda), \infty) \times \hat{\mathscr{K}}$. Then using the strong maximum principle, we argue that if (μ, Θ) lies in it, then $\Theta'(\chi) < 0$ on $[\frac{1}{4}\lambda, \frac{1}{2}\lambda]$. The significance of this observation, which lies in the fact that Θ represents the angle of inclination of the free surface (suitably parametrized) with the horizontal, is discussed, and the possibility of extending the method to get information about the shape of the extreme wave is mentioned, but no firm conclusion is reached.† Using an idea of Benjamin, we show that the maximum angle of inclination of any periodic or solitary water-wave under consideration (those in the sets $\mathscr{C}_\lambda$ or $\mathscr{C}'$ in theorems 2.2 and 3.5, respectively) is less than $\frac{1}{3}\pi$. (These conclusions are described, more plainly, in §2.4.)

† See footnote on p. 649.

Finally, the main result of this paper is proved in § 3, and is summarized as follows: if h is fixed, then as $\lambda \to \infty$ the connected sets of periodic waves of wavelength λ on a flow of mean depth h converge, in a certain sense, to a connected set of solitary waves whose asymptotic height is h. This connected set enjoys all the properties of the connected set $\mathscr{C}$ mentioned in I, theorem 3.9, and the behaviour of the corresponding waves is described in I, § 4. The global existence of solitary waves is already known (I); what is new here is that periodic waves converge to solitary waves in the long-wave limit. An easy corollary of our general result in this direction is the following:

Corollary 3.4. *For each β, $0 < \beta < \frac{1}{6}\pi$, and $h, \lambda > 0$ there exists on a flow of mean depth h, a periodic, symmetric water-wave of wavelength λ, the free surface of which subtends an angle β with the horizontal at its steepest point. If h is fixed and $\lambda_n \uparrow \infty$ as $n \to \infty$, then a subsequence of the periodic wave profiles converge uniformly on compact subsets of $\mathbb{R}$ to the profile of a steady solitary water-wave whose free surface subtends a maximum angle of β with the horizontal, and whose asymptotic depth is h.*

Such results as these may be regarded as global versions of the theorems of Ter-Krikerov (1960, 1963) and Lavrentiev (1943, 1947, 1954) who proved existence of small-amplitude solitary waves by showing the convergence of small-amplitude periodic waves to solitary waves as their wavelength increases indefinitely. (See Bona *et al.* for another global treatment of a related problem.) It is worth noting that because the mathematical theory of large-amplitude water-waves lacks any form of global uniqueness result, we cannot claim that all solitary waves may be described as the long-wave limit of a sequence of periodic waves. The results of § 3 follow immediately by the methods of I, § 3 once the similarity between (1.31) and equation (3.10) of I has been noted. The analysis presented here has the advantage that the linearization about the zero solution of (1.31) is well understood because exact solutions can be found. It may be regarded as a small step towards finding theoretical confirmation of the very striking numerical results given by Cokelet (1977).

1.2. *The water-wave problems*

The question being considered is the existence problem for steady, two-dimensional waves on the surface of an ideal liquid acted on by gravity. In this section two possible types of flow are considered.

(*a*) *A symmetric, periodic flow of wavelength λ whose mean depth is h*†

If such a flow exists and if the free surface has a unique maximum per wavelength, then a cross-section of the flow perpendicular to the wave crests may be identified with a region in the complex z-plane between the line $y = 0$ and a curve $\{x + iH_\lambda(x) \colon x \in \mathbb{R}\}$. Here $H_\lambda \colon \mathbb{R} \to (0, \infty)$ is a function of period λ which is even and is decreasing on the interval $(0, \frac{1}{2}\lambda)$ (see figure 1). One wavelength of this flow then occupies the region S_λ bounded by the lines $x = \pm\frac{1}{2}\lambda, y = 0$ and the free surface $\Gamma_\lambda = \{x + iH_\lambda(x) \colon x \in (-\frac{1}{2}\lambda, \frac{1}{2}\lambda)\}$. Since the fluid is supposed to be incompressible and the flow irrotational, there exists an analytic function, the complex potential, $\omega = \phi + i\psi$, which is related to the velocity $(u(z), v(z))$ of the flow at a point $z \in S_\lambda$ by the expression

$$u(z) - iv(z) = -d\omega/dz = -\phi_x + i\phi_y = -\psi_y - i\psi_x. \tag{1.1}$$

† The mean depth h is defined in (1.9) by $h = Q/c$, where c is the mean velocity defined by (1.6) and Q is the flux carried by the flow. The definition of mean velocity is one of two which Stokes (1847, pp. 444–445) considered as being reasonable for the water-wave problem. The mean depth is sometimes called the undisturbed depth (Cokelet 1977). (See also Wehausen & Laitone 1960, pp. 456–457.)

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

Since the flow is symmetric about $x = 0$, ω must satisfy the relation

$$\overline{\frac{d\omega}{dz}(z)} = \frac{d\omega}{dz}(-\bar{z}), \tag{1.2}$$

whence

$$\psi_x(z) = -\psi_x(-\bar{z}) \tag{1.3}$$

and

$$\psi(z) = \psi(-\bar{z}). \tag{1.4}$$

In particular, ψ_x is zero on the imaginary axis and, by periodicity,

$$\psi_x(z) = -\phi_y(z) = 0 \quad \text{if} \quad \operatorname{Re} z = \pm \tfrac{1}{2}\lambda. \tag{1.5}$$

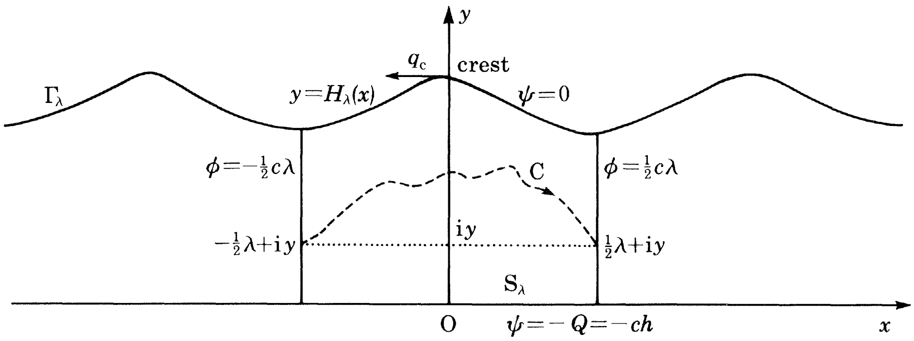

FIGURE 1. A steady wave of wavelength λ on a flow of mean depth h. The region occupied by one wavelength is S_λ. The mean velocity of the flow is $-c$, and its speed at the crest is q_c.

Let $C = \{z(t): t \in [0, 1]\}$ be any simple curve in S_λ directed from $-\frac{1}{2}\lambda + iy$ to $\frac{1}{2}\lambda + iy$. Then

$$\begin{aligned}\int_C \{u(z) - iv(z)\}\, dz &= \omega(-\tfrac{1}{2}\lambda + iy) - \omega(\tfrac{1}{2}\lambda + iy) \\ &= \phi(-\tfrac{1}{2}\lambda + iy) - \phi(\tfrac{1}{2}\lambda + iy),\end{aligned}$$

by (1.4),

$$= -(\phi(\tfrac{1}{2}\lambda) - \phi(-\tfrac{1}{2}\lambda))$$

by (1.5). In particular, if C is chosen to be a horizontal line (for example, the bottom of the domain S_λ), we find that

$$\frac{1}{\lambda}\int_C u(z)\, dz = \frac{1}{\lambda}\int_C \{u(z) + iv(z)\}\, dz = -\{\phi(\tfrac{1}{2}\lambda) - \phi(-\tfrac{1}{2}\lambda)\}/\lambda, \tag{1.6}$$

which is called the *mean velocity* and is denoted by $-c$. (If the flow is considered in a frame of reference relative to which the mean velocity is zero, then c is the phase speed of the wave.) Since the bottom $(y = 0)$ and the free surface Γ_λ are streamlines, the stream-function ψ must be constant on both, and without loss of generality, we may suppose that

$$\psi(z) = 0 \quad \text{if} \quad z \in \Gamma_\lambda. \tag{1.7}$$

Since h is the *mean depth* of the flow,

$$\psi(z) = -Q \quad \text{if} \quad \operatorname{Im} z = 0, \tag{1.8}$$

where

$$Q = ch. \tag{1.9}$$

(Note that for a given flow the mean depth is not to be confused with an integral average of the height of the free surface. It is defined by (1.9) once the *flux* Q of the flow is known. By definition,

$-Q$ is the value of ψ on the bottom when ψ has been normalized so that $\psi = 0$ on the free surface.) Finally, since Γ_λ is a *free streamline*, the pressure is a constant there, and Bernoulli's theorem then implies that

$$\tfrac{1}{2}|\nabla\phi(z)|^2 + g\,\mathrm{Im}\,z = \text{const.} \tag{1.10}$$

for all $z \in \Gamma_\lambda$, where g is the acceleration due to gravity.

The existence question for this type of periodic flow is first one of finding the region S_λ occupied by one wavelength of the flow, and then one of finding ϕ and ψ such that a periodic flow of wavelength λ and mean depth h occupies S_λ. It must be shown that ϕ and ψ satisfy all the conditions (1.1)–(1.5), (1.7), (1.8) and (1.10) in S_λ, where Q is given by (1.9) and c is given by (1.6).

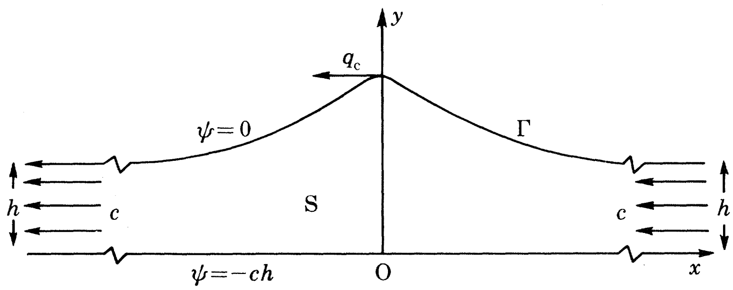

FIGURE. 2. The region occupied by a steady solitary wave of asymptotic velocity c (from right to left) and asymptotic height h.

(b) *Solitary waves on a flow of asymptotic depth h*

By a steady solitary wave is meant a symmetric two-dimensional flow whose free surface is in the form of a single symmetric wave of elevation, whose extent is infinite, and which is asymptotic to a finite height at $\pm\infty$ (see figure 2). The flow at $\pm\infty$ is supposed to be approximately uniform horizontal flow from right to left in the channel. The boundary-value problem posed by this situation is first to find the flow domain S bounded by the line $y = 0$ and a curve $\Gamma = \{x + iH(x): x \in \mathbb{R}\}$, where the even function H is decreasing on $(0, \infty)$ and

$$\lim_{|x|\to\infty} H(x) = h, \tag{1.11}$$

and then to find a complex potential ω satisfying all the boundary conditions, which, in this case, take the following form. The relation between the complex potential and the velocity field is given by (1.1), and since the flow is symmetrical (1.2) must also be satisfied. Since the flow is supposed approximately uniform and horizontal at points of S far from the crest, we have

$$\lim_{|z|\to\infty} u(z) - iv(z) = \lim_{|z|\to\infty} -\frac{d\omega}{dz}(z) = -c \quad (z \in S), \tag{1.12}$$

where $-c$ is the asymptotic velocity of the steady flow. (In a frame of reference relative to which the asymptotic speed is zero, c is the phase speed of the wave.) Since Γ and the bottom are both streamlines, we may suppose that

$$\psi = 0 \quad \text{on} \quad \Gamma \tag{1.13}$$

and

$$\psi = -ch \quad \text{if} \quad \mathrm{Im}\,z = 0. \tag{1.14}$$

The boundary condition (1.13) is a normalization, as before, and (1.14) follows from (1.11) because the stream function is a constant on the bottom, and by (1.12), $\psi_y(z) \to c$ as $|z| \to \infty$, $z \in S$. Finally, since Γ is a *free* streamline, Bernoulli's theorem requires that (1.10) must hold on Γ.

If, by analogy with the periodic case, the mean velocity of a solitary wave is calculated from the formula $\lim_{\lambda\to\infty} \lambda^{-1}\int_{-\frac{1}{2}\lambda}^{\frac{1}{2}\lambda} u(x+iy)\,dx$ for any $y \in (0, h)$, then it follows from (1.12) that this value coincides with the asymptotic velocity. Since the flux of the solitary wave is ch, it follows that the mean depth and the asymptotic depth coincide.

It is appropriate at this stage to mention one further parameter associated with a flow, solitary or periodic. In either case, q_c is used to denote the *speed of the steady flow at the crest* of a wave.

As is well known, both the free boundary-value problems described above can be reformulated as nonlinear integral equations (Milne-Thomson 1968; Nekrasov 1967). In § 1.4 this formulation is discussed in detail, but before we can do that we need to introduce some conformal mappings. The approach of the next two sections is suggested by the remarks in the appendix of Keady *et al.* (1978).

1.3. *Preliminary mapping theorems*

A treatment of Jacobi's elliptic functions which is adequate for our purposes is given in Copson (1972). The reader may also refer to Nehari (1952) for additional material. Throughout this section, and in all of the sequel, h is an arbitrary but *fixed* positive real number.

For $k \in (0, 1)$, $\mathrm{sn}(\cdot, k)$ denotes the odd Jacobi elliptic function of modulus k with primitive periods $4K$ and $2iK'$ and simple poles with residues $1/k$ or $-1/k$ at points congruent to iK' or to $2K+iK'$ (mod $4K$, $2iK'$), respectively. The primitive periods are given in terms of k by the formulae

$$K = \int_0^1 (1-x^2)^{-\frac{1}{2}}(1-k^2x^2)^{-\frac{1}{2}}\,dx, \tag{1.15}$$

and

$$K' = \int_0^1 (1-x^2)^{-\frac{1}{2}}(1-(1-k^2)\,x^2)^{-\frac{1}{2}}\,dx. \tag{1.16}$$

Clearly K and K' are monotone functions of $k \in (0, 1)$, with $K \downarrow \frac{1}{2}\pi$ and $K' \uparrow \infty$ as $k \downarrow 0$, while $K \uparrow \infty$ and $K' \downarrow \frac{1}{2}\pi$ as $k \uparrow 1$.

For any $\lambda > 0$, let k_λ be the modulus of the unique function $\mathrm{sn}(\cdot, k_\lambda)$ such that

$$\lambda/h = 4K_\lambda/K'_\lambda, \tag{1.17}$$

where K_λ and K'_λ are defined in terms of k_λ by (1.15) and (1.16). Since h is fixed, k_λ, K_λ and K'_λ are monotone functions of λ and

$$2K_\lambda/\lambda \to \pi/4h \tag{1.18}$$

as $\lambda \to \infty$. Throughout we shall use s_λ to denote the elliptic function $\mathrm{sn}(\cdot, k_\lambda)$ for all $\lambda > 0$, and s_∞ to denote the analytic function tanh which is the pointwise limit of s_λ as $\lambda \to \infty$ (Copson 1972, p. 414, ex. 1).

Standard theory (Copson 1972, p. 414, ex. 4; or Kober 1952, p. 172) ensures that the mapping $\tilde{p}_\lambda$ from the complex ζ-plane into the complex ξ-plane defined by

$$\tilde{p}_\lambda(\zeta) = -k_\lambda s_\lambda^2(2K_\lambda(\zeta+ih)/\lambda)$$

is a conformal mapping of the region $R_\lambda = \{\zeta = \chi + i\eta\colon -\frac{1}{2}\lambda < \chi < \frac{1}{2}\lambda,\ -h < \eta < 0\}$ onto the region $\mathscr{D}' = \{\xi = r\,e^{is}\colon 0 < r < 1,\ -\pi < s < \pi\}$. The function $\tilde{p}_\lambda$ is analytic on $\bar{R}_\lambda$ and maps the boundary portion $A_\lambda = \{\zeta \in \bar{R}_\lambda\colon \zeta = \chi + i0,\ \chi \in [-\frac{1}{2}\lambda, \frac{1}{2}\lambda)\}$ onto the set $\{\xi = e^{is}\colon -\pi < s \leqslant \pi\}$, and maps $\partial R_\lambda \backslash A_\lambda$ onto the non-positive real axis in the unit disc.† Let $p_\lambda: [-\frac{1}{2}\lambda, \frac{1}{2}\lambda) \to (-\pi, \pi]$ be defined as follows:

$$p_\lambda(\chi) = s \quad \text{for} \quad \chi \in [-\tfrac{1}{2}\lambda, \tfrac{1}{2}\lambda),$$

if and only if $s \in (-\pi, \pi]$ and

$$\tilde{p}_\lambda(\chi + \mathrm{i}0) = \mathrm{e}^{\mathrm{i}s}.$$

Another, more convenient way of saying this is that for $\chi \in [-\frac{1}{2}\lambda, \frac{1}{2}\lambda)$, $p_\lambda(\chi) = s$ if and only if $s \in (-\pi, \pi]$ and

$$\begin{aligned}\cos\tfrac{1}{2}s + \mathrm{i}\sin\tfrac{1}{2}s &= -\mathrm{i}k_\lambda^{\frac{1}{2}} s_\lambda(2K_\lambda(\chi + \mathrm{i}h)/\lambda) \\ &= -\mathrm{i}\left\{\frac{(1+k_\lambda)\, s_\lambda(2K_\lambda\chi/\lambda) + \mathrm{i}c_\lambda(2K_\lambda\chi/\lambda)\, d_\lambda(2K_\lambda\chi/\lambda)}{1 + k_\lambda s_\lambda^2(2K_\lambda\chi/\lambda)}\right\},\end{aligned} \tag{1.19}$$

where c_λ and d_λ denote the even Jacobi elliptic functions cn $(\cdot, k_\lambda)$ and dn $(\cdot, k_\lambda)$, respectively. (Algebraic identities and rules for differentiating the functions c_λ, d_λ and s_λ are given in Copson (1972, p. 384).) The expression (1.19) follows from the relation (1.17) and Copson (1972, p. 396, example 3).)‡ Let $\tilde{q}_\lambda: \mathscr{D}' \to R_\lambda$ denote the inverse of $\tilde{p}_\lambda$, and let $q_\lambda: (-\pi, \pi] \to [-\frac{1}{2}\lambda, \frac{1}{2}\lambda)$ denote the inverse of p_λ. From equation (1.19) it follows that

$$\sin\tfrac{1}{2}s = -\frac{(1+k_\lambda)\, s_\lambda(2K_\lambda q_\lambda(s)/\lambda)}{1 + k_\lambda s_\lambda^2(2K_\lambda q_\lambda(s)/\lambda)},$$

which, upon differentiating with respect to s and using (1.19) along with the identities in Copson (1972, p. 384), yields

$$\tfrac{1}{2}\cos\tfrac{1}{2}s = -\frac{2K_\lambda(1+k_\lambda)}{\lambda}\left\{\frac{1 - k_\lambda s_\lambda^2(2K_\lambda q_\lambda(s)/\lambda)}{1 + k_\lambda s_\lambda^2(2K_\lambda q_\lambda(s)/\lambda)}\right\} q_\lambda'(s)\cos\tfrac{1}{2}s \tag{1.20}$$

where $'$ denotes differentiation. But, by the algebraic identities relating s_λ, c_λ and d_λ there results that

$$(1 - k_\lambda s_\lambda^2)^2 = c_\lambda^2 d_\lambda^2 + (1 - k_\lambda)^2 s_\lambda^2,$$

and so (1.19) and (1.20) together yield the following expression for q_λ':

$$q_\lambda'(s) = -\{\lambda/(4K_\lambda(1+k_\lambda))\}\left[\cos^2\tfrac{1}{2}s + \left(\frac{1-k_\lambda}{1+k_\lambda}\right)^2 \sin^2\tfrac{1}{2}s\right]^{-\frac{1}{2}}. \tag{1.21}$$

To simplify the notation, we define the following expressions:

$$\left.\begin{aligned} f_\lambda(s) &= \frac{1}{2}\left[\cos^2\tfrac{1}{2}s + \left(\frac{1-k_\lambda}{1+k_\lambda}\right)^2 \sin^2\tfrac{1}{2}s\right]^{-\frac{1}{2}}, \\ f(s) &= \tfrac{1}{2}\sec\tfrac{1}{2}s, \end{aligned}\right\} \tag{1.22}$$

for all $s \in (-\pi, \pi)$, and

$$\varLambda = \lambda/(2K_\lambda(1+k_\lambda)). \tag{1.23}$$

Recall from (1.18) that

$$\varLambda \to 2h/\pi \quad \text{as} \quad \lambda \to \infty. \tag{1.24}$$

Since the only zeros of $\mathrm{d}\tilde{p}_\lambda/\mathrm{d}\zeta$ occur at $\zeta = -\mathrm{i}h$ and at $\zeta = \pm\frac{1}{2}\lambda - \mathrm{i}h$, the real and imaginary parts of $\tilde{q}_\lambda$ satisfy the Cauchy–Riemann conditions on the boundary portion $\{\mathrm{e}^{\mathrm{i}t}: -\pi < t < \pi\}$ of $\partial\mathscr{D}'$. Hence

$$\begin{aligned}\frac{\partial}{\partial r}(\operatorname{Im}\tilde{q}_\lambda)\bigg|_{\mathrm{e}^{\mathrm{i}s}} &= -\frac{\partial}{\partial s}(\operatorname{Re}\tilde{q}_\lambda)\bigg|_{\mathrm{e}^{\mathrm{i}s}} = -q_\lambda'(s) \\ &= \varLambda f_\lambda(s)\end{aligned} \tag{1.25}$$

for all $s \in (-\pi, \pi)$.

† The minus sign in the definition of $\tilde{p}_\lambda$ reverses the usual orientation and causes the point $\zeta = -\frac{1}{2}\lambda + \mathrm{i}0$ to be mapped onto $\xi = \mathrm{e}^{\mathrm{i}\pi}$.

‡ The negative root is taken in (1.19) since the point $s = \pi$ corresponds to $\chi = -\frac{1}{2}\lambda$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Before finishing this discussion of conformal mappings, we note that in the limiting case when $\lambda \to \infty$ a mapping which takes the region $R_\infty = \{\chi + i\eta: \chi \in (-\infty, \infty), -h < \eta < 0\}$ conformally on to $\mathscr{D}'$ and the boundary portion $A_\infty = \{\chi + i0: \chi \in (-\infty, \infty)\}$ onto $\{e^{is}: -\pi < s < \pi\}$, is given by

$$\tilde{p}(\zeta) = -\tanh^2(\pi(\zeta + ih)/4h)$$
$$= -s_\infty^2(\pi(\zeta + ih)/4h).$$

If the inverse of $\tilde{p}$ is denoted by $\tilde{q}$, then it follows just as before that

$$\frac{\partial}{\partial r}(\operatorname{Im} \tilde{q})\Big|_{e^{is}} = -\frac{\partial}{\partial s}(\operatorname{Re} \tilde{q})\Big|_{e^{is}}$$
$$= (h/\pi) \sec \tfrac{1}{2}s = (2h/\pi) f(s). \tag{1.26}$$

If $v: [-\pi, \pi] \to \mathbb{R}$ is a continuous, odd function with $v(\pi) = 0$, then there exists a unique harmonic function $\tilde{u}$ on the unit disc $\mathscr{D} = \{\xi: |\xi| < 1\}$ which satisfies the Neumann boundary condition $\partial \tilde{u}/\partial r|_{e^{is}} = v(s), s \in (-\pi, \pi]$, and the normalization condition $\int_{\partial \mathscr{D}} \tilde{u} = 0$. It is easy to see that for all $s \in (-\pi, \pi]$,

$$\tilde{u}(e^{is}) = \int_{-\pi}^{\pi} G(s, t)\, v(t)\, dt,$$

where

$$G(s, t) = \frac{1}{\pi} \sum_{l=1}^{\infty} \frac{\sin ls \sin lt}{l}$$
$$= \frac{1}{2\pi} \ln \left| \frac{\sin \frac{1}{2}(s+t)}{\sin \frac{1}{2}(s-t)} \right| \tag{1.27}$$

for all $(s, t) \in (-\pi, \pi] \times (-\pi, \pi], s \neq t$, and $\tilde{u}$ is zero on the real axis in $\mathscr{D}$. (The identity (1.27) and further properties of G are from I, theorem 2.5.) Note that (1.27) ensures that $G(s, t) \geqslant 0$ for all $(s, t) \in [0, \pi] \times [0, \pi], s \neq t$.

The next theorem concerns the change of variables which enables the convergence result of §3 to be deduced from the work of I.

THEOREM 1.1. *Let* $V: [-\frac{1}{2}\lambda, \frac{1}{2}\lambda) \to \mathbb{R}$ *be a continuous, odd function which is positive on* $(0, \frac{1}{2}\lambda)$ *with* $V(-\frac{1}{2}\lambda) = 0$. *Then putting* $v(s) = -V(q_\lambda(s))$, *for all* $s \in (-\pi, \pi]$, *defines a continuous, odd function which is positive on* $(0, \pi)$, *and* $v(\pi) = 0$.

Moreover, if Λ *is given by* (1.23), *then*

$$u(s) = \frac{\Lambda}{2\pi} \int_{-\pi}^{\pi} \ln \left| \frac{\sin \frac{1}{2}(s+t)}{\sin \frac{1}{2}(s-t)} \right| f_\lambda(t)\, v(t)\, dt \tag{1.28}$$

for all $s \in (-\pi, \pi]$, *if and only if* $\quad u(s) = -U(q_\lambda(s))$,

where

$$U(\chi) = \int_{-\frac{1}{2}\lambda}^{\frac{1}{2}\lambda} \frac{1}{2\pi} \ln \left| \frac{s_\lambda(2K_\lambda(\chi + \epsilon)/\lambda)}{s_\lambda(2K_\lambda(\chi - \epsilon)/\lambda)} \right| V(\epsilon)\, d\epsilon. \tag{1.29}$$

Furthermore, there exists a harmonic function $\tilde{U}$ *on* R_λ *such that*

$$\tilde{U}(\chi + i0) = U(\chi) \quad \text{for} \quad \chi \in [-\tfrac{1}{2}\lambda, \tfrac{1}{2}\lambda),$$
$$\frac{\partial \tilde{U}}{\partial \eta}\Big|_{\chi + i0} = V(\chi) \quad \text{for} \quad \chi \in (-\tfrac{1}{2}\lambda, \tfrac{1}{2}\lambda),$$

and $\tilde{U} = 0$ *on* $\partial R_\lambda \backslash A_\lambda$.

Proof. It follows from (1.19) and from the formula for the elliptic function of a sum that, under the change of variables

$$\chi = q_\lambda(s) \quad \text{and} \quad \epsilon = q_\lambda(t), \quad s, t \in (-\pi, \pi],$$

the kernel

$$\frac{1}{2\pi} \ln \left| \frac{s_\lambda(2K_\lambda(\chi+\epsilon)/\lambda)}{s_\lambda(2K_\lambda(\chi-\epsilon)/\lambda)} \right|$$

becomes

$$\frac{1}{2\pi} \ln \left| \frac{\sin \frac{1}{2}(s+t)}{\sin \frac{1}{2}(s-t)} \right|.$$

Since $q'_\lambda(s) = -\Lambda f_\lambda(s)$ on $(-\pi, \pi]$, the result for the first part of the theorem is immediate.

Because v is continuous and odd on $(-\pi, \pi]$ and $v(\pi) = 0$, it follows that there exists a unique function $\tilde{u}$, harmonic on $\mathscr{D}$ and continuous on $\overline{\mathscr{D}}$, such that

$$\left. \frac{\partial \tilde{u}}{\partial r} \right|_{e^{it}} = \Lambda f_\lambda(t)\, v(t)$$

and

$$\tilde{u}(e^{it}) = u(t)$$

for all $t \in (-\pi, \pi]$. Since v is odd, $\tilde{u}$ is zero on the real axis in $\mathscr{D}$. Therefore $\tilde{U}$ defined by

$$\tilde{U}(\zeta) = -\tilde{u}(\tilde{p}_\lambda(\zeta))$$

is harmonic on R_λ and continuous on $\bar{R}_\lambda$. Since $\tilde{p}_\lambda$ maps $\partial R_\lambda \backslash A_\lambda$ onto the non-positive real axis in $\mathscr{D}$, where $\tilde{u}$ vanishes, it follows that $\tilde{U}$ vanishes on $\partial R_\lambda \backslash A_\lambda$. The results for $\tilde{U}$ on A_λ follow by (1.25).

Lemma 1.2. *For all* $\chi, \epsilon \in [-\frac{1}{2}\lambda, \frac{1}{2}\lambda] \times [-\frac{1}{2}\lambda, \frac{1}{2}\lambda]$, $\chi \neq \epsilon$,

$$\frac{1}{2\pi} \ln \left| \frac{s_\lambda(2K_\lambda(\chi+\epsilon)/\lambda)}{s_\lambda(2K_\lambda(\chi-\epsilon)/\lambda)} \right| = \frac{1}{\pi} \sum_{l=1}^{\infty} l^{-1} \tanh\left(\frac{2\pi l h}{\lambda}\right) \sin\left(\frac{2\pi l \chi}{\lambda}\right) \sin\left(\frac{2\pi l \epsilon}{\lambda}\right).$$

Proof. This follows by a simple calculation from the expansion from sn (u, k) (Gradshteyn & Ryzhik 1965, p. 912, eqn. 20):

$$\ln \operatorname{sn}(u, k) = \ln \frac{2K}{\pi} + \ln \sin \frac{\pi u}{2K} - 4 \sum_{l=1}^{\infty} \frac{1}{l} \frac{q^l}{1+q^l} \left\{ \sin\left(\frac{\pi l u}{2K}\right) \right\}^2$$

where $q = e^{-\pi K'/K}$.

Theorem 1.3. *The solutions of the linear characteristic value problem*

$$u(s) = \tfrac{1}{3}\mu \int_{-\pi}^{\pi} G(s,t) f_\lambda(t)\, u(t)\, dt$$

consist precisely of the set of characteristic values $\{(6\Lambda\pi l/\lambda) \coth(2\pi l h/\lambda)\}_{l=1}^{\infty}$ *with corresponding eigenfunctions* $\{\sin(2\pi l q_\lambda(s)/\lambda)\}_{l=1}^{\infty}$. *In particular, the smallest characteristic value,* $\mu_\lambda = (6\Lambda\pi/\lambda) \coth(2\pi h/\lambda) \downarrow 6/\pi$ *as* $\lambda \to \infty$.

Proof. From lemma 1.2 it follows that the set of characteristic values of the operator defined by the right-hand side of (1.29) comprise the set $\{(2\pi l/\lambda) \coth(2\pi l h/\lambda)\}_{l=1}^{\infty}$, and the corresponding eigenvectors are $\{\sin(2\pi l \chi/\lambda)\}_{l=1}^{\infty}$. The result is then an immediate consequence of theorem 1.1 and the fact that $\Lambda \to 2h/\pi$ as $\lambda \to \infty$ by (1.24). Since $f_{\lambda_1}(x) \leqslant f_{\lambda_2}(x)$ if $\lambda_1 \leqslant \lambda_2$, it follows that $\mu_{\lambda_1} \geqslant \mu_{\lambda_2}$ by I, theorems A 1 and A 2. (See also lemma 3.3.)

1.4. *On integral equations for water-waves*

The purpose of this section is to show the equivalence of two nonlinear integral equations, each of which is a formulation of the periodic water-wave problem when the mean depth and the wavelength are given. Theorem 1.4 is a statement of this equivalence, while in theorem 1.5 a precise description of the wave which corresponds to a solution of equation (1.31) is given. Theorem 1.6, which is taken without proof from I, is a statement of the corresponding result for solitary waves.

Let h be fixed, as in the previous section, and let λ be any positive real number.

THEOREM 1.4. (i) *If* $\Theta\colon [-\frac{1}{2}\lambda, \frac{1}{2}\lambda) \to \mathbb{R}$ *is continuous, odd, and* $0 < \Theta(\chi) < \frac{1}{2}\pi$ *on* $(0, \frac{1}{2}\lambda)$, *with* $\Theta(-\frac{1}{2}\lambda) = 0$, *and if for all* $s \in (-\pi, \pi]$,

$$\theta(s) = -\Theta(q_\lambda(s)), \tag{1.30}$$

then $\theta\colon (-\pi, \pi] \to \mathbb{R}$ *is continuous, odd, and* $0 < \theta(s) < \frac{1}{2}\pi$ *on* $(0, \pi)$ *and* $\theta(\pi) = 0$. *Moreover, for some* $\mu > 0$, θ *satisfies the equation*

$$\theta(s) = \frac{1}{6}\int_{-\pi}^{\pi} \frac{1}{\pi} \ln\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right| \frac{f_\lambda(t)\sin\theta(t)}{\dfrac{1}{\mu} + \displaystyle\int_0^t f_\lambda(w)\sin\theta(w)\,\mathrm{d}w}\,\mathrm{d}t \tag{1.31}$$

for all $s \in (-\pi, \pi]$, *if and only if* Θ *satisfies the equation*

$$\Theta(\chi) = \frac{1}{6}\int_{-\frac{1}{2}\lambda}^{\frac{1}{2}\lambda} \frac{1}{\pi} \ln\left|\frac{s_\lambda(2K_\lambda(\chi+\epsilon)/\lambda)}{s_\lambda(2K_\lambda(\chi-\epsilon)/\lambda)}\right| \frac{\sin\Theta(\epsilon)}{\dfrac{\Lambda}{\mu} + \displaystyle\int_0^\epsilon \sin\Theta(w)\,\mathrm{d}w}\,\mathrm{d}\epsilon \tag{1.32}$$

for all $\chi \in [-\frac{1}{2}\lambda, \frac{1}{2}\lambda)$. *Here* Λ *is given by* (1.23) *and* f_λ *by* (1.22).

(ii) *If* Θ *is as in* (i) *and satisfies* (1.32), *then there exists a harmonic function on* R_λ *which coincides with* Θ *on the boundary portion* A_λ, *and which is zero on* $\partial R_\lambda \backslash A_\lambda$. *If* $\tilde{\Theta}$ *is used to denote this harmonic function on* R_λ, *then*

$$\left.\frac{\partial\tilde{\Theta}}{\partial\eta}\right|_{\chi+\mathrm{i}0} = \frac{1}{3}\frac{\sin\Theta(\chi)}{\dfrac{\Lambda}{\mu} + \displaystyle\int_0^\chi \sin\Theta(w)\,\mathrm{d}w} \tag{1.33}$$

for all $\chi + \mathrm{i}0 \in A_\lambda = [-\frac{1}{2}\lambda, \frac{1}{2}\lambda)$.

Proof. The first part follows immediately from the definition of q_λ as the inverse of p_λ. The equivalence of (1.31) and (1.32) may be seen by changing variables and by using (1.25).

The next result is a precise statement of the sense in which solutions of equation (1.31) correspond to non-trivial periodic water-waves. Once the method of I has been applied to prove the convergence of solutions of the equation for waves of period λ to solitary wave solutions as $\lambda \to \infty$, the convergence of periodic waves to solitary waves in the physical domain will follow.

THEOREM 1.5. *Suppose that* θ *is an odd, continuous function on* $[-\pi, \pi]$ *with* $0 < \theta(s) \leqslant \pi$ *on* $(0, \pi)$ *and* $\theta(\pi) = 0$, *which satisfies the integral equation* (1.31) *on* $[-\pi, \pi]$ *for some* $\mu > 0$. *Then* θ *is real-analytic on* $[-\pi, \pi]$ *and satisfies* $0 < \theta(s) < \frac{1}{2}\pi$ *on* $(0, \pi)$. *Moreover, there exists a solution of the periodic water-wave problem of period* λ *on a flow of mean depth* h. *The mean velocity of the flow is given by*

$$c = \frac{\surd(3g)}{\Lambda}\left(\frac{2}{\lambda}\int_0^\pi \frac{f_\lambda(t)\cos\theta(t)}{\left\{\dfrac{1}{\mu} + \displaystyle\int_0^t f_\lambda(w)\sin\theta(w)\,\mathrm{d}w\right\}^{\frac{1}{3}}}\,\mathrm{d}t\right)^{-\frac{3}{2}}, \tag{1.34}$$

from which the flux Q and the speed at the crest q_c may be calculated as follows:

$$Q = ch \tag{1.35}$$

and
$$q_c = (3gAc/\mu)^{\frac{1}{3}}. \tag{1.36}$$

The free surface Γ_λ is then given by $\{(x, H_\lambda(x)) : x \in (-\frac{1}{2}\lambda, \frac{1}{2}\lambda)\}$, where for $x \in (0, \frac{1}{2}\lambda)$,

$$H_\lambda(x) - H_\lambda(0) = \left(\frac{c^2 \Lambda^2}{3g}\right)^{\frac{1}{3}} \int_{\alpha^{-1}(x)}^{0} \frac{f_\lambda(t)\sin\theta(t)}{\left(\frac{1}{\mu} + \int_0^t f_\lambda(w)\sin\theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}t, \tag{1.37}$$

and for $s \in (-\pi, 0)$,

$$\alpha(s) = \left(\frac{c^2 \Lambda^2}{3g}\right)^{\frac{1}{3}} \int_{s}^{0} \frac{f_\lambda(t)\cos\theta(t)}{\left(\frac{1}{\mu} + \int_0^t f_\lambda(w)\sin\theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}t. \tag{1.38}$$

Remark. In this expression for the free surface profile, the value of $H_\lambda(0)$ is given by $\int_{-h}^{0} \exp(\tilde{T}(0+\mathrm{i}\eta))\,\mathrm{d}\eta$, were $\tilde{T}$ is the function in (1.39) below. Since $\tilde{T}$ is uniquely determined in R_λ by $\tilde{\theta}$ and (1.39), one may determine $H_\lambda(0)$ explicitly in terms of μ, θ, h, and λ. Unfortunately, it does not appear possible to put this result in as neat a form as (1.34) (1.37), or (1.38).

In theorem 2.4, we show that the upper bound of $\frac{1}{2}\pi$ for θ may be replaced by $\frac{1}{3}\pi$.

Proof. The method of proof of theorem 2.2 (ii), (iii) applies to any solution of (1.31), and not just to those in $\mathscr{C}_\lambda$. Hence, the real-analyticity of θ and the *a priori* bound of $\frac{1}{2}\pi$ for θ follow immediately.

Let $\tilde{\Theta}$ be the function which is harmonic on R_λ mentioned in theorem 1.4 (ii), and $\tilde{T}$ denote the unique function which is harmonic on R_λ and conjugate to $\tilde{\Theta}$ (that is, $\tilde{T} - \mathrm{i}\tilde{\Theta}$ is analytic in R_λ) such that

$$\frac{1}{\lambda}\int_{-\frac{1}{2}\lambda}^{\frac{1}{2}\lambda} \exp(\tilde{T}(\chi - \mathrm{i}h))\,\mathrm{d}\chi = 1. \tag{1.39}$$

If $\tilde{\theta}$ denotes the harmonic function on $\mathscr{D}$ with boundary values θ, then the real-analyticity of θ ensures that $\tilde{\theta}$ is real-analytic on $\overline{\mathscr{D}}$. Since $\tilde{\Theta}(\zeta) = -\tilde{\theta}(\tilde{p}_\lambda(\zeta))$, $\zeta \in \bar{R}_\lambda$, the analyticity of $\tilde{p}_\lambda$ on $\bar{R}_\lambda$ ensures that $\tilde{\Theta}$ is real-analytic on $\bar{R}_\lambda$, and hence that $\tilde{T}$ is real-analytic by the Cauchy–Riemann equations. Because $\tilde{T} - \mathrm{i}\tilde{\Theta}$ is analytic on $\bar{R}_\lambda$, we can use it to define an analytic function $\tilde{m}$ on $\bar{R}_\lambda$ by putting

$$\tilde{m}(\zeta) = \int_{-\mathrm{i}h}^{\zeta} \exp\{\tilde{T}(\xi) - \mathrm{i}\tilde{\Theta}(\xi)\}\,\mathrm{d}\xi. \tag{1.40}$$

The function $\tilde{m}$ is injective on $\bar{R}_\lambda$; for otherwise there exist $\zeta_1, \zeta_2 \in \bar{R}_\lambda$ with

$$\int_{\zeta_1}^{\zeta_2} \exp(\tilde{T}(\xi))\cos\tilde{\Theta}(\xi)\,\mathrm{d}\xi = 0,$$

and this contradicts the fact that $|\tilde{\Theta}| < \frac{1}{2}\pi$ on $\bar{R}_\lambda$ by the maximum principle. Since $\tilde{m}'(\zeta) \neq 0$ in $\bar{R}_\lambda$, it follows that R_λ is mapped conformally onto a region S_λ by $\tilde{m}$, and that $\tilde{m}$ is invertible there.

Because $\tilde{\Theta}$ is odd on A_λ, and zero on the rest of ∂R_λ, it follows that $\overline{\tilde{\Theta}(\zeta)} = -\tilde{\Theta}(-\bar{\zeta})$ and $\tilde{T}(\zeta) = \tilde{T}(-\bar{\zeta})$, $\zeta \in \bar{R}_\lambda$. From this observation and (1.40) there results that $\overline{-\tilde{m}(\zeta)} = \tilde{m}(-\bar{\zeta})$, $\zeta \in \bar{R}_\lambda$. Combining this with (1.39) and the fact that $\tilde{\Theta} = 0$ on $\partial R_\lambda \backslash A_\lambda$ yields that S_λ is bounded by the lines $y = 0$, $x = \pm\frac{1}{2}\lambda$ and the curve $\Gamma_\lambda = \tilde{m}(A_\lambda)$. If ζ lies on the line A_λ, then

$$(\mathrm{d}/\mathrm{d}\chi)\,\mathrm{Re}\,\tilde{m}(\zeta) = \exp(\tilde{T}(\zeta))\cos\tilde{\Theta}(\zeta) > 0,$$

and so, for some even function H_λ, we have

$$\Gamma_\lambda = \{x+\mathrm{i}H_\lambda(x) : x\in[-\tfrac{1}{2}\lambda, \tfrac{1}{2}\lambda)\}.$$

A further calculation based on (1.40) yields that

$$H'_\lambda(x) = -\tan\Theta(\tilde{m}^{-1}(x+\mathrm{i}H_\lambda(x))).$$

Since $\tilde{m}$ is invertible, $\tilde{m}^{-1}$ is analytic on $\bar{\mathrm{S}}_\lambda$. We shall now show that if a complex potential ω is defined on $\bar{\mathrm{S}}_\lambda$ by putting

$$\omega(z) = \phi(z)+\mathrm{i}\psi(z) = c\tilde{m}^{-1}(z) \tag{1.41}$$

with c given by (1.34), then all of the conditions (1.1)–(1.10) are satisfied, and the proof of the theorem will be complete.

The velocity field (u, v) generated in $\bar{\mathrm{S}}_\lambda$ by ω is given by

$$\begin{aligned} u(z)-\mathrm{i}v(z) &= -\mathrm{d}\omega/\mathrm{d}z \\ &= -c\exp(-\tilde{T}(\tilde{m}^{-1}(z)))\{\cos\tilde{\Theta}(\tilde{m}^{-1}(z))+\mathrm{i}\sin\tilde{\Theta}(\tilde{m}^{-1}(z))\}, \end{aligned} \tag{1.42}$$

whence $-\tilde{\Theta}(\tilde{m}^{-1}(z))$ is the angle which the negative velocity vector makes with the x-axis, and $c\exp(-\tilde{T}(\tilde{m}^{-1}(z)))$ is the speed of the flow at $z\in\bar{\mathrm{S}}_\lambda$. Since $-\overline{\tilde{m}(\zeta)} = \tilde{m}(-\bar{\zeta})$, $\zeta\in\bar{R}_\lambda$, it follows that equations (1.2)–(1.4) are satisfied. Since $\tilde{\Theta}=0$ on $\partial R_\lambda\backslash A_\lambda$, it is immediate from (1.40) that (1.5) holds. To show that (1.6) is satisfied we note that, by (1.39),

$$\begin{aligned} \lambda^{-1}\{\phi(-\tfrac{1}{2}\lambda)-\phi(\tfrac{1}{2}\lambda)\} &= c\lambda^{-1}\operatorname{Re}\{\tilde{m}^{-1}(-\tfrac{1}{2}\lambda)-\tilde{m}^{-1}(\tfrac{1}{2}\lambda)\} \\ &= c\lambda^{-1}\{-\tfrac{1}{2}\lambda-\tfrac{1}{2}\lambda\} = -c. \end{aligned} \tag{1.43}$$

Next, if $z\in\Gamma_\lambda$, then $\tilde{m}^{-1}(z)\in A_\lambda$, whence $\psi(z) = 0$. If $z\in\bar{\mathrm{S}}_\lambda$ and $\operatorname{Im} z = 0$, then $\operatorname{Im}\tilde{m}^{-1}(z) = -h$, and so $\psi(z) = -ch$. It follows that (1.7)–(1.9) hold.

Let $T: [-\tfrac{1}{2}\lambda, \tfrac{1}{2}\lambda]\to\mathbb{R}$ denote the restriction of $\tilde{T}$ to A_λ. Then, since $\tilde{\Theta} = 0$ on $\partial R_\lambda\backslash A_\lambda$, it follows, by Cauchy's theorem and (1.39), that

$$\int_{-\frac{1}{2}\lambda}^{\frac{1}{2}\lambda}\exp(T(\chi))\cos\Theta(\chi)\,\mathrm{d}\chi = \int_{-\frac{1}{2}\lambda}^{\frac{1}{2}\lambda}\exp(\tilde{T}(\chi-\mathrm{i}h))\,\mathrm{d}\chi = \lambda. \tag{1.44}$$

However, from (1.33) and the Cauchy–Riemann equations,

$$T(\chi) = T(0)-\tfrac{1}{3}\ln\left(1+\left(\frac{\mu}{\Lambda}\right)\int_0^\chi\sin\Theta(w)\,\mathrm{d}w\right) \tag{1.45}$$

for all $\chi\in[-\tfrac{1}{2}\lambda, \tfrac{1}{2}\lambda]$. Substituting this expression for T into (1.44) gives

$$\begin{aligned} \exp(-T(0)) &= \frac{1}{\lambda}\int_{-\frac{1}{2}\lambda}^{\frac{1}{2}\lambda}\frac{\cos\Theta(\chi)}{\left(1+\left(\frac{\mu}{\Lambda}\right)\int_0^\chi\sin\Theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\mathrm{d}\chi \\ &= \frac{1}{\lambda}\int_{-\pi}^{\pi}\frac{\Lambda f_\lambda(t)\cos\theta(t)}{\left(1+\mu\int_0^t f_\lambda(w)\sin\theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\mathrm{d}t = (3g\Lambda/\mu c^2)^{\frac{1}{3}}, \end{aligned} \tag{1.46}$$

by (1.34), whence $$q_c^3 = c^3\exp(-3T(0)) = 3g\Lambda c/\mu,$$

and so (1.36) is satisfied.

In order to prove (1.10), we must show that $\frac{1}{2}(u^2(z)+v^2(z))+g\,\mathrm{Im}\,z$ is constant on Γ_λ, or, equivalently, that $\frac{1}{2}c^2\exp(-2T(\chi))+g\,\mathrm{Im}\,\tilde{m}(\chi+\mathrm{i}0)$ is constant for $\chi\in[-\frac{1}{2}\lambda,\frac{1}{2}\lambda]$. A calculation gives

$$\begin{aligned}(\mathrm{d}/\mathrm{d}\chi)\,[\tfrac{1}{2}c^2\exp(-2T(\chi))+g\,\mathrm{Im}\,\tilde{m}(\chi+\mathrm{i}0)]\\ &= -c^2\exp(-2T(\chi))\,T'(\chi)-g\exp(T(\chi))\sin\Theta(\chi)\\ &= \exp(T(\chi))\{-c^2\exp(-3T(\chi))T'(\chi)-g\sin\Theta(\chi)\}\\ &= 0\end{aligned}$$

by (1.45) and (1.46).

Finally, to calculate the wave profile we proceed as follows. At a point $x+\mathrm{i}y\in\Gamma_\lambda$, the free surface is given by

$$y = H_\lambda(x),$$

where $H'_\lambda(x) = -\tan\Theta(\tilde{m}^{-1}(x+\mathrm{i}H_\lambda(x)))$. Hence

$$\begin{aligned}H_\lambda(x)-H_\lambda(0) &= \int_0^x H'_\lambda(w)\,\mathrm{d}w\\ &= -\int_0^{\tilde{\alpha}^{-1}(x)}\tan\Theta(\chi)\cos\Theta(\chi)\exp(T(\chi))\,\mathrm{d}\chi\\ &= -\left(\frac{\mu c^2}{3g\Lambda}\right)^{\frac{1}{3}}\int_0^{\tilde{\alpha}^{-1}(x)}\frac{\sin\Theta(\chi)}{\left(1+\left(\frac{\mu}{\Lambda}\right)\int_0^\chi\sin\Theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}\chi,\end{aligned}$$

where $\tilde{\alpha}\colon[-\frac{1}{2}\lambda,\frac{1}{2}\lambda)\to\mathbb{R}$ is given by

$$\begin{aligned}\tilde{\alpha}(\chi) &= \mathrm{Re}\,\tilde{m}(\chi+\mathrm{i}0)\\ &= \left(\frac{\mu c^2}{3g\Lambda}\right)^{\frac{1}{3}}\int_0^\chi\frac{\cos\Theta(\chi')}{\left(1+\left(\frac{\mu}{\Lambda}\right)\int_0^{\chi'}\sin\Theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}\chi'.\end{aligned}$$

Hence if $x\in[0,\frac{1}{2}\lambda]$,

$$H_\lambda(x)-H_\lambda(0) = \left(\frac{\Lambda^2c^2}{3g}\right)^{\frac{1}{3}}\int_{\alpha^{-1}(x)}^0\frac{f_\lambda(t)\sin\theta(t)}{\left(\frac{1}{\mu}+\int_0^t f_\lambda(w)\sin\theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}t$$

where $\quad \alpha^{-1}(x) = (\tilde{\alpha}\circ q_\lambda)^{-1}(x) = p_\lambda(\tilde{\alpha}^{-1}(x)) < 0,$

and so, for $s\in(-\pi,0]$,

$$\begin{aligned}\alpha(s) = \tilde{\alpha}\circ q_\lambda(s) &= \left(\frac{\mu c^2}{3g\Lambda}\right)^{\frac{1}{3}}\int_0^{q_\lambda(s)}\frac{\cos\Theta(\chi')}{\left(1+\left(\frac{\mu}{\Lambda}\right)\int_0^{\chi'}\sin\Theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}\chi'\\ &= \left(\frac{\Lambda^2c^2}{3g}\right)^{\frac{1}{3}}\int_s^0\frac{f_\lambda(t)\cos\theta(t)}{\left(\frac{1}{\mu}+\int_0^t f_\lambda(w)\sin\theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}t.\end{aligned}$$

This completes the proof of the theorem.

Theorem 1.6. *Suppose that θ is an odd, continuous function on $[-\pi,\pi]$ with $0<\theta(s)\leqslant\pi$ on $(0,\pi)$ and $\theta(\pi)=0$, which satisfies the integral equation*

$$\theta(s) = \frac{1}{6}\int_{-\pi}^{\pi}\frac{1}{\pi}\ln\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right|\frac{f(t)\sin\theta(t)}{\frac{1}{\mu}+\int_0^t f(w)\sin\theta(w)\,\mathrm{d}w}\,\mathrm{d}t \qquad (1.47)$$

for all $s \in [-\pi, \pi]$, *where* $\mu > 0$ *and* $f(t) = \frac{1}{2}\sec\frac{1}{2}t$ *for* $t \in (-\pi, \pi)$. *Then* $\theta f \in L_1(-\pi, \pi)$, θ *is real-analytic on* $(-\pi, \pi)$, *and* $0 < \theta(s) < \frac{1}{2}\pi$ *on* $(0, \pi)$. *Moreover, if h and c are any positive real numbers which satisfy*

$$\frac{6gh}{\pi c^2}\left(\frac{1}{\mu}+\int_0^\pi f(w)\sin\theta(w)\,\mathrm{d}w\right) = 1,$$

then there exists a steady solitary wave flow whose mean velocity† *and asymptotic height (see* §1.2*) are* $-c$ *and* h *respectively. The speed* q_c *of the flow at the wave crest may be calculated from the expression*

$$\pi q_c^3/6ghc = 1/\mu.$$

Moreover, the solitary wave profile Γ *is given by* $\{(x, H(x)) : x \in \mathbb{R}\}$ *where for* $x > 0$,

$$H(x)-H(0) = \frac{h}{3}\left(\frac{36c^2}{\pi^2 gh}\right)^{\frac{1}{3}}\int_{\alpha^{-1}(x)}^{0}\frac{f(t)\sin\theta(t)}{\left(\dfrac{1}{\mu}+\displaystyle\int_0^t f(w)\sin\theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}t \tag{1.48}$$

and for $s \in (-\pi, 0)$,

$$\alpha(s) = \frac{h}{3}\left(\frac{36c^2}{\pi^2 gh}\right)^{\frac{1}{3}}\int_{s}^{0}\frac{f(t)\cos\theta(t)}{\left(\dfrac{1}{\mu}+\displaystyle\int_0^t f(w)\sin\theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}t. \tag{1.49}$$

Remark. In this case, we can assert that the value of $H(0)$ is

$$h\left\{1+\frac{1}{2}\frac{c^2}{gh}-\frac{1}{2}\left(\frac{36c^2}{\pi^2\mu^2 gh}\right)^{\frac{1}{3}}\right\},$$

because the asymptotic height is known (see I, theorem 4.6).

Proof. While this theorem is formally the limiting case of theorem 1.5 as $\lambda \to \infty$, it needs a separate proof. This may be done by modifying the method of proof of theorem 1.5, using the mapping $\tilde{p}$ from R_∞ onto $\mathscr{D}'$ introduced in §1.3. The function Θ is then required to be in $L_1(-\infty, \infty)$, odd, positive on $(0, \infty)$, and to satisfy

$$\Theta(\chi) = \frac{1}{6}\int_{-\infty}^{\infty}\frac{1}{\pi}\ln\left|\frac{\tanh(\pi(\chi+\epsilon)/4h)}{\tanh(\pi(\chi-\epsilon)/4h)}\right|\frac{\sin\Theta(\epsilon)}{\dfrac{2h}{\pi\mu}+\displaystyle\int_0^\epsilon\sin\Theta(w)\,\mathrm{d}w}\,\mathrm{d}\epsilon, \tag{1.50}$$

an equation which may be obtained from equation (1.47) by putting $\chi = (-2h/\pi)\ln(\sec\frac{1}{2}s+\tan\frac{1}{2}s)$, and $\epsilon = (-2h/\pi)\ln(\sec\frac{1}{2}t+\tan\frac{1}{2}t)$, $s, t \in (-\pi, \pi)$. An alternative proof is to be found in I, theorems 1.1, 4.1, 4.3 and 4.6. (The function Θ in I, theorem 1.1 differs from that which arises in the method suggested by the proof of theorem 1.5 by a change of sign.)

For the sake of giving a complete description of Nekrasov's integral equations, we include in the Appendix the equation for periodic waves on a flow which is infinitely deep. The derivation there is slightly different from those already in the literature, and emphasizes the dependence of the flow parameters on a given solution of the equation. It is shown how this equation can be written in an alternative form which involves the conjugate operator from the L_2-theory of Fourier series. While a similar formulation might be adopted in the case of finite depth (Krasovskii 1961), we avoid this approach because the normalization requirement ((1.39) above and

† Recall from the Introduction that in the case of solitary waves, the notion of mean velocity coincides with that of asymptotic velocity, and similarly, the mean depth and the asymptotic depth coincide.

Krasovskii (1961, p. 1002, eqn (1.19)) means that when the depth is finite the conjugate operator is nonlinear. In any case, (1.31) and (1.32) are preferred, since the dependence of the integrand on θ and Θ is given explicitly.

2. The global theory

2.1. *Background*

The first proof of the existence of large amplitude, periodic water-waves is due to Krasovskii (1961) and is based on an adaptation of the monotone minorant theorem (Krasnosel'skii 1964) to a particular version of Nekrasov's equation. Among his results on the existence of periodic water-waves in a channel with a wave-like bottom is included the special case when the bottom is flat. In this case, the conclusion is that *for each positive h and* λ, *and for each* $\beta \in (0, \frac{1}{6}\pi)$, *there exists a wave of wavelength* λ, *on a flow whose mean depth is h, which is such that the maximum angle of inclination of the free surface to the horizontal is* β *and the mean velocity of all such waves is bounded away from zero and infinity.* Though this result is highly suggestive, it does not amount to a global bifurcation theorem since neither the question of bifurcation, nor the question of the existence of a connected set of solutions is considered. The first result of this kind is due to Keady *et al.* (1978), who regard Nekrasov's integral equation as an example in the general theory of global bifurcation (Dancer 1973; Rabinowitz 1971; Turner 1975). They proved the following: *if L and Q are fixed positive real numbers, then there exists a connected set of periodic water-waves which bifurcates from the set of horizontal, uniform flows, each of which is of flux Q, and each of which has wavelength 2L with respect to the velocity potential. This set contains a wave whose speed at the crest is* q_c *for any value of* q_c *in the interval* $(0, (gL\pi^{-1}\tanh(\pi Q/L))^{\frac{1}{3}})$.

Since the mathematical theory of steady water-waves still lacks any global uniqueness result, it is *not* possible to assert that the solutions obtained by Krasovskii are included in the connected set which Keady and Norbury obtain. (In principle, Krasovskii's method may yield solutions lying off the bifurcating set, if such exist.) Nevertheless, it can be shown (Toland 1978, independently of the work of Krasovskii) that this bifurcating set contains waves with maximum angle of inclination to the horizontal β, for all values of β in the interval $(0, \frac{1}{6}\pi)$. Indeed, it has been shown by McLeod (1982) that this connected set of water-waves contains a wave whose maximum angle of inclination to the horizontal is β, for all $\beta \in (0, \frac{1}{6}\pi + \epsilon]$ for some $\epsilon > 0$.

In the next section, we shall summarize the global bifurcation theory for periodic water-waves *of spatial wavelength* λ *on a flow of mean depth h.* Because of our declared intention to deduce from these results the corresponding theorems for solitary waves on a flow of mean depth h, we state theorems about the periodic problem in terms of the integral equation (1.31) rather than the equivalent equation (1.32). In §2.3, we shall see how the use of (1.32) leads to new results about the bifurcation of periodic waves, which are obscured by the formulation of the problem as (1.31).

2.2. *The bifurcation of periodic waves of wavelength* λ *on a flow of mean depth h*

Throughout this section, we consider waves of wavelength λ on a flow of *fixed* mean depth h. Accordingly, we are interested in solutions (μ, θ) of (1.31) with $\mu > 0$ and $0 < \theta(s) < \frac{1}{2}\pi$ on $(0, \pi)$. Since all solutions of (1.31) are odd, it suffices instead to consider the eigenvalue problem

$$\theta(s) = \frac{2}{3}\int_0^\pi G(s,t)\,\frac{f_\lambda(t)\sin\theta(t)}{\dfrac{1}{\mu} + \displaystyle\int_0^t f_\lambda(w)\sin\theta(w)\,\mathrm{d}w}\,\mathrm{d}t \tag{2.1}$$

where the kernel G is defined in (1.27). Let $C_0[a,b]$ denote the Banach space of continuous functions on $[a,b]$ which vanish at a and b, and let $\mathscr{K}_0[a,b]$ denote the closed, reproducing cone of non-negative functions in $C_0[a,b]$. For any $[a,b] \subset [0,\pi], C[a,b]$ denotes the usual Banach space of continuous functions on $[a,b]$ with the supremem norm. For convenience with notation, we will abbreviate $\mathscr{K}_0[0,\pi]$ as $\mathscr{K}_0$. Since G is non-negative almost everywhere on $[0,\pi]\times[0,\pi]$ and is the kernel of a compact, linear Hammerstein operator on $C_0[0,\pi]$ (I, theorem 2.5 (*a*), (*b*)), it follows that this linear operator leaves $\mathscr{K}_0$ invariant. The linearization of (2.1) about $\theta = 0$ is given by

$$\theta(s) = \frac{2\mu}{3}\int_0^\pi G(s,t) f_\lambda(t)\,\theta(t)\,\mathrm{d}t, \tag{2.2}$$

and from theorem 1.3 it follows that the characteristic value with smallest absolute value is $6\Lambda\pi\lambda^{-1}\coth(2\pi h/\lambda) \to 6/\pi$ as $\lambda\to\infty$, and the corresponding eigenvector is $\sin(2\pi q_\lambda(s)/\lambda)$. Before the global bifurcation result may be stated, one further observation is necessary.

Lemma 2.1. *Let $\mu > 0$, and let $\theta \in \mathscr{K}_0$ be such that, for all $s\in[0,\pi]$,*

$$\theta(s) = \frac{2}{3}\int_0^\pi G(s,t)\,\frac{f_\lambda(t)\sin(J\theta(t))}{\dfrac{1}{\mu}+\displaystyle\int_0^t f_\lambda(w)\sin(J\theta(w))\,\mathrm{d}w}\,\mathrm{d}t, \tag{2.3}$$

where $\quad Jx = (\operatorname{sgn} x)\min\{|x|,\pi\},\quad$ *for all* $\quad x\in\mathbb{R}.$

Then (i) $0<\theta(s)<\pi$ *on* $(0,\pi)$, *and* (ii) $\mu > \mu_\lambda = 6\Lambda\pi\lambda^{-1}\coth(2\pi h/\lambda)$.

Proof. The proof of this result is an easy consequence of the maximum principle, and is proved by the method used to establish theorem 3.3 (*a*), (*c*) of I. No modifications are required.

The next result is a summary of the global existence theory for solutions of equation (1.31). Throughout the discussion, the mean depth is fixed. Let $\mathscr{S}_\lambda = \{(\mu,\theta)\in(0,\infty)\times\mathscr{K}_0\colon (\mu,\theta)$ satisfies (2.1) and $\theta\neq 0\}\cup\{(\mu_\lambda,0)\}$ where $\mu_\lambda = 6\Lambda\pi\lambda^{-1}\coth(2\pi h/\lambda)$. Section 2.3 gives more sophisticated properties of $\mathscr{S}_\lambda$; in particular, the upper bound of $\frac{1}{2}\pi$ in (ii) and (vi) may be replaced by $\frac{1}{3}\pi$. Much of this result is well-known, though maybe in a different form. We outline the proof for completeness.

Theorem 2.2. *Let $\mathscr{C}_\lambda$ denote the maximal connected subset of $\mathscr{S}_\lambda$ in $\mathbb{R}\times C_0[0,\pi]$ which contains $(\mu_\lambda,0)$. Then*

(i) *$\mathscr{C}_\lambda$ is closed and unbounded.*

(ii) *If $(\mu,\theta)\in\mathscr{C}_\lambda\backslash\{(\mu_\lambda,0)\}$, then $\mu>\mu_\lambda$ and $0<\theta(s)<\frac{1}{2}\pi$ on $(0,\pi)$, whence $\{\mu\colon(\mu,\theta)\in\mathscr{C}_\lambda\} = [\mu_\lambda,\infty)$.*

(iii) *θ is a real-analytic function on $[0,\pi]$.*

(iv) *For each $\lambda,\delta>0$, there exists a constant $B_{\lambda,\delta}>0$ such that*

$$\theta(s) \geqslant B_{\lambda,\delta}\sin s \tag{2.4}$$

if $\mu>\mu_\lambda+\delta$ and $(\mu,\theta)\in\mathscr{C}_\lambda$.

(v) *If $(\mu,\theta)\in\mathscr{C}_\lambda$, then the mean velocity of the corresponding wave is given by the formula (1.34) and will be denoted by $c(\mu,\theta)$. For each $\lambda>0$, there exists a closed interval $[a_\lambda,b_\lambda]\subset(0,\infty)$ such that*

$$\{c(\mu,\theta)\colon(\mu,\theta)\in\mathscr{C}_\lambda\}\subset[a_\lambda,b_\lambda],$$

and $\quad a_\lambda\to 0$ *as* $\lambda\to 0$, *while* $b_\lambda\leqslant M$ *for all* $\lambda>0$,

where M is independent of λ.

Let the speed of the corresponding flow at the wave crest, calculated from (1.36), *be denoted by* $q_c(\mu, \theta)$.

(vi) *If* $\{(\mu_n, \theta_n)\} \subset \mathscr{C}_\lambda$ *and* $\mu_n \to \infty$ *as* $n \to \infty$, *then* $q_c(\mu_n, \theta_n) \to 0$, *and there exists a subsequence* $\{\theta_{n(k)}\}$ *of* $\{\theta_n\}$ *such that* $\theta_{n(k)} \to \theta$ *uniformly on* $[\delta, \pi]$ *for each* $\delta > 0$, *where* θ *is a non-trivial solution of the equation*

$$\theta(s) = \frac{2}{3}\int_0^\pi G(s,t)\frac{f_\lambda(t)\sin\theta(t)}{\int_0^t f_\lambda(w)\sin\theta(w)\,dw}\,dt \quad \textit{for} \quad s \in (0, \pi]. \tag{2.5}$$

The function θ *is real-analytic on* $(0, \pi]$ *and* $0 < \theta(s) < \frac{1}{2}\pi$ *on* $(0, \pi)$. *Furthermore the following dichotomy holds:*† *either* $\lim_{s\to 0+} \theta(s) = \frac{1}{6}\pi$, *or*

$$0 < \liminf_{s\to 0+} \theta(s) < \tfrac{1}{6}\pi < \limsup_{s\to 0+} \theta(s).$$

The periodic wave corresponding to a solution of (2.5) *has a stagnation point at its crest* (i.e. $q_c = 0$).

(vii) *Let* $\{(\mu_n, \theta_n)\} \subset \mathscr{C}_\lambda$ *denote the subsequence in* (vi). *Since* (μ_n, θ_n) *satisfies equation* (2.1), *it follows that the function* θ_n^* *defined on* $[0, \mu_n \pi]$ *by*

$$\theta_n^*(x) = \theta_n(x/\mu_n),$$

satisfies the equation

$$\theta_n^*(x) = \frac{2}{3}\int_0^{\mu_n\pi} G\left(\frac{x}{\mu_n}, \frac{y}{\mu_n}\right)\frac{f_\lambda(y/\mu_n)\sin\theta_n^*(y)}{1 + \int_0^y f_\lambda(w/\mu_n)\sin\theta_n^*(w)\,dw}\,dy$$

for all $x \in [0, \mu_n \pi]$.

Moreover, as $n \to \infty$, $\{\theta_n^*\}$ *converges uniformly on compact subsets of* $(0, \infty)$ *to a function* θ^* *which satisfies the boundary-layer equation*

$$\theta^*(x) = \frac{2}{3}\int_0^\infty \frac{1}{2\pi}\ln\left|\frac{x+y}{x-y}\right| \frac{\frac{1}{2}\sin\theta^*(y)}{1 + \frac{1}{2}\int_0^y \sin\theta^*(w)\,dw}\,dy,$$

and $\sup_{x\in(0,\infty)} \theta^*(x) > \frac{1}{6}\pi$. *It follows that there exists an* $\epsilon > 0$ *such that, for all* n *sufficiently large,* $|\theta_n|_{C_0[0,\pi]} \geqslant \frac{1}{6}\pi + \epsilon$. *Hence, for each* $\beta \in [0, \frac{1}{6}\pi + \epsilon]$, *there exists a periodic water wave of any specified mean depth and wavelength, the free surface of which subtends a maximum angle to the horizontal of* β.

(viii) *For each* $N > 0$, *the set* $\{(\mu, \theta) \in \mathscr{C}_\lambda \colon \mu \leqslant N\}$ *is relatively compact in the topology of* $\mathbb{R} \times C^l$ *for each integer* $l \geqslant 0$, *where* C^l *is the Banach space of* lth *order continuously differentiable functions on* $[0, \pi]$.

Proof. (i) The proof of this is a simple application of Dancer (1973, theorem 2) to equation (2.3), once the *a priori* bound of lemma 2.1 has been noted (see Keady *et al.* (1978, lemma 4.1) for a similar treatment of equation (1.32)).

(ii) That $\mu > \mu_\lambda$ follows after multiplying equation (2.1) by f_λ and by the eigenfunction of the linear equation (2.2), which corresponds to the characteristic value μ_λ, and integrating over $(0, \pi)$.

A slight modification of I, theorem 3.3 (d) yields that $\theta(s) < \frac{1}{2}\pi$ on $(0, \pi)$. In this case the crucial observation is that the function P defined on $\mathscr{D}'$ by putting

$$P(\zeta) = -\tfrac{1}{2}\exp(-2\tilde{\rho}(\zeta)) - Y(\zeta)$$

is a super-harmonic function on $\mathscr{D}'$ which attains its minimum at every point of the boundary portion $\{e^{it}\colon t \in (-\pi, \pi]\}$. (The use of the super-harmonic pressure function P to show that the

† Professor L. E. Fraenkel and the authors have now proved that $\theta(s) \to \frac{1}{6}\pi$ as $s \to 0+$ for solutions of equation (2.5) when $0 < \lambda \leqslant \infty$. (See Amick, Fraenkel & Toland 1982.)

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

free surface of periodic water-waves have no vertical tangents was introduced by Spielvogel, (1970), and used again by Keady *et al.* (1978).) Here $\tilde{\rho}$ and Y are defined as follows. If $(\mu, \theta) \in \mathscr{C}_\lambda$ then suppose that

$$\theta(s) = \sum_{l=1}^{\infty} a_l \sin ls,$$

and put
$$\rho(t) = -\tfrac{1}{3}\ln\left(\frac{1}{\mu} + \int_0^t f_\lambda(w)\sin\theta(w)\,\mathrm{d}w\right)$$

for $t \in [-\pi, \pi]$. If $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \rho(t)\,\mathrm{d}t$, then it follows that

$$F(r\,\mathrm{e}^{\mathrm{i}t}) = a_0 + \sum_{l=1}^{\infty} a_l r^l \mathrm{e}^{\mathrm{i}lt}$$

for $r \in [0, 1)$, $t \in (-\pi, \pi]$ defines an analytic function on $\mathscr{D}$. Then put

$$\tilde{\rho}(\zeta) = \operatorname{Re} F(\zeta),$$

and
$$Y(\zeta) = \operatorname{Im} \frac{1}{3\Lambda}\int_0^{\zeta} \exp\left(F(\xi)\right) \tilde{q}_\lambda'(\xi)\,\mathrm{d}\xi$$

for $\zeta \in \mathscr{D}'$ where $\tilde{q}_\lambda$ is the inverse of the conformal mapping $\tilde{p}_\lambda$ introduced in §1.2 (and prime denotes differentiation). With this definition of P, the proof that $\theta < \frac{1}{2}\pi$ follows exactly as in I, theorem 3.3 (*d*).

(iii) Lewy's (1952) theorem ensures that θ is real-analytic on $[0, \pi]$.

(iv) If this result is false, then for some $\delta > 0$ and for each n, there exists $(\mu_n, \theta_n) \in \mathscr{C}_\lambda \cap \{[\mu_\lambda + \delta, \infty) \times \mathscr{K}_0\}$ and $s_n \in (0, \pi)$ such that $\theta_n(s_n) \leqslant n^{-1} \sin s_n$. Now for each closed interval $[a, b] \subset (0, \pi)$, there exists $E > 0$ (depending on $[a, b]$) such that if $t \in [a, b]$, then

$$G(s, t) \geqslant E \sin s$$

for all $s \in [0, \pi]$ (see I, theorem 2.5 (*c*)). Hence

$$n^{-1}\sin s_n \geqslant \theta_n(s_n) = \frac{2}{3}\int_0^{\pi} G(s_n, t)\,\frac{f_\lambda(t)\sin\theta_n(t)}{\dfrac{1}{\mu_n} + \displaystyle\int_0^t f_\lambda(w)\sin\theta_n(w)\,\mathrm{d}w}\,\mathrm{d}t$$
$$\geqslant \left\{\frac{2E}{3}\int_a^b \frac{f_\lambda(t)\sin\theta_n(t)}{\dfrac{1}{\mu_n} + \displaystyle\int_0^t f_\lambda(w)\sin\theta_n(w)\,\mathrm{d}w}\,\mathrm{d}t\right\}\sin s_n.$$

Since $[a, b]$ is chosen arbitrarily in $(0, \pi)$, there results that

$$\frac{f_\lambda(t)\sin\theta_n(t)}{\dfrac{1}{\mu_n} + \displaystyle\int_0^t f_\lambda(w)\sin\theta_n(w)\,\mathrm{d}w} \to 0,$$

almost everywhere in $[0, \pi]$. From the *a priori* bound established in (ii), it follows that $\theta_n \to 0$ in $L_1(0, \pi)$. However, an integration of (2.1) over $(0, \pi)$ after multiplication by $\sin s$ yields that

$$\int_0^{\pi} \theta_n(s)\sin s\,\mathrm{d}s = \frac{1}{3}\int_0^{\pi} \frac{f_\lambda(t)\sin\theta_n(t)\sin t}{\dfrac{1}{\mu_n} + \displaystyle\int_0^t f_\lambda(w)\sin\theta_n(w)\,\mathrm{d}w}\,\mathrm{d}t \geqslant \frac{1}{3\pi}\,\frac{\displaystyle\int_0^{\pi}\theta_n(t)\sin t\,\mathrm{d}t}{\dfrac{1}{\mu_n} + \displaystyle\int_0^{\pi} f_\lambda(w)\sin\theta_n(w)\,\mathrm{d}w},$$

whence $\{\mu_n\}$ is bounded, since $f_\lambda \sin\theta_n \to 0$ in $L_1(0,\pi)$. Because G is the kernel of a compact linear operator on $C_0[0,\pi]$, and because $\{\mu_n\}$ is bounded, it follows that θ_n converges to 0 in $C_0[0,\pi]$. But $\mathscr{C}_\lambda$ is closed, from which there follows the contradiction that the sequence $\{\mu_n\} \subset [\mu_\lambda+\delta,\infty)$ converges to μ_λ.

(v) If $(\mu,\theta)\in\mathscr{C}_\lambda$, $\lambda>0$, then by (1.23) and (1.34) there results that

$$c(\mu,\theta) \leqslant \text{const.}\times K_\lambda \lambda^{\frac{1}{2}} \left\{\int_0^\pi \frac{f_\lambda(t)\cos\theta(t)}{\left(\dfrac{1}{\mu}+\displaystyle\int_0^t f_\lambda(w)\sin\theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}t\right\}^{-\frac{3}{2}}.$$

Hence, for any $N>0$, the set $\{c(\mu,\theta)\colon (\mu,\theta)\in\mathscr{C}_\lambda, \lambda\in(0,N]\}$ is bounded above, or else there exists a sequence $(\mu_n,\theta_n)\in\mathscr{C}_{\lambda_n}$, $\lambda_n\in(0,N]$, such that

$$\int_0^\pi \frac{f_{\lambda_n}(t)\cos\theta_n(t)}{\left(\dfrac{1}{\mu_n}+\displaystyle\int_0^t f_{\lambda_n}(w)\sin\theta_n(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}t \to 0$$

as $n\to\infty$.

In the latter case it follows, by the bounds in (ii) above, that $\theta_n \to \frac{1}{2}\pi$ in $L_1(0,\pi)$, and $\sin\theta_n\to 1$ in $L_1(0,\pi)$. Without loss of generality, suppose that $1/\mu_n \to \alpha\in[0,\frac{1}{6}\pi]$ and $\lambda_n\to\lambda\in[0,N]$ as $n\to\infty$. Hence, for any interval $[a,b]\subset(0,\pi)$, it follows by I, theorem 2.5 (c) that

$$\theta_n(s) \geqslant \frac{2}{3}\int_a^b G(s,t)\,\frac{f_{\lambda_n}(t)\sin\theta_n(t)}{\dfrac{1}{\mu_n}+\displaystyle\int_0^t f_{\lambda_n}\sin\theta_n}\,\mathrm{d}t \geqslant \text{const.}\times\left\{\int_a^b \frac{f_{\lambda_n}(t)\sin\theta_n(t)}{\dfrac{1}{\mu_n}+\displaystyle\int_0^t f_{\lambda_n}\sin\theta_n}\,\mathrm{d}t\right\}\sin s \geqslant \text{const.}\times\sin s,$$

where the constants are independent of sufficiently large n. Hence, by the dominated convergence theorem

$$\ln\left(\frac{1}{\mu_n}+\int_0^t f_{\lambda_n}\sin\theta_n\right) \to \ln\left(\alpha+\int_0^t f_\lambda\right)$$

in $L_1(0,\pi)$, as $n\to\infty$, and so for any $l\geqslant 1$,

$$\int_0^\pi \sin ls\,\theta_n(s)\,\mathrm{d}s = \frac{1}{3l}\int_0^\pi \frac{\sin lt f_{\lambda_n}(t)\sin\theta_n(t)}{\dfrac{1}{\mu_n}+\displaystyle\int_0^t f_{\lambda_n}\sin\theta_n}\,\mathrm{d}t = -\frac{1}{3}\int_0^\pi \cos lt \ln\left(\frac{1}{\mu_n}+\int_0^t f_{\lambda_n}\sin\theta_n\right)\mathrm{d}t$$

$$\to -\frac{1}{3}\int_0^\pi \cos lt\ln\left(\alpha+\int_0^t f_\lambda\right)\mathrm{d}t \tag{2.6}$$

as $n\to\infty$.

Therefore, for each integer $l\geqslant 1$, equation (2.6) gives

$$\tfrac{1}{2}\pi\int_0^\pi \sin ls\,\mathrm{d}s = -\frac{1}{3}\int_0^\pi \cos lt\ln\left(\alpha+\int_0^t f_\lambda\right)\mathrm{d}t \tag{2.7}$$

$$= \frac{1}{3l}\int_0^\pi \sin lt\,\frac{f_\lambda(t)}{\alpha+\displaystyle\int_0^t f_\lambda}\,\mathrm{d}t.$$

However this is a contradiction since, if α is non-zero the right-hand side is $o(1/l)$ by the Riemann–Lebesgue lemma while the left-hand side is not (for odd l). In the case $\alpha=0$ the right-hand side may be re-written as

$$\frac{1}{3l}\int_0^\pi \frac{\sin lt}{t}\,\mathrm{d}t+\frac{1}{3l}\int_0^\pi\left\{\frac{f_\lambda(t)}{\displaystyle\int_0^t f_\lambda}-\frac{1}{t}\right\}\sin lt\,\mathrm{d}t \sim \frac{\pi}{6l}+o(1/l) \quad \text{as} \quad l\to\infty$$

while the left-hand side of (2.7) vanishes for all even l.

Hence the set $\{c(\mu,\theta): (\mu,\theta)\in\mathscr{C}_\lambda, \lambda\in(0,N]\}$ is bounded above. In order to show that an upper bound may be found which is independent of N, we proceed as before by seeking a contradiction. If the result is false, then $c(\mu_n,\theta_n)\to\infty$ for some sequence $\{(\mu_n,\theta_n)\}$, where $(\mu_n,\theta_n)\in\mathscr{C}_{\lambda_n}$ and $\lambda_n\to\infty$. However, a slight modification of the proof of theorem 3.1 (iv) yields that there must therefore exist a subsequence $\{(\mu_{n(k)},\theta_{n(k)})\}$ such that $(1/\mu_{n(k)},\theta_{n(k)})\to(\alpha,\theta)\in[0,\infty)\times L_2(0,\pi)$, and $c(\mu_{n(k)},\theta_{n(k)})\to\{[6gh/\pi]\,(\alpha+\int_0^\pi f(t)\sin\theta(t)\,\mathrm{d}t)\}^{\frac{1}{2}}\in[\surd(gh), 2\surd(gh)]$. This is a contradiction.

Finally, to show that, for fixed λ, the set $\{c(\mu,\theta): (\mu,\theta)\in\mathscr{C}_\lambda\}$ is bounded below by a positive constant, it suffices to observe that

$$\begin{aligned} c(\mu,\theta) &\geqslant \text{const.}\times\left\{\int_0^\pi\left(\frac{1}{\mu}+\int_0^t f_\lambda(w)\sin\theta(w)\,\mathrm{d}w\right)^{-\frac{1}{3}}\mathrm{d}t\right\}^{-\frac{3}{2}} \\ &\geqslant \text{const.}\quad(\text{by (iv)}) \end{aligned}$$

where both constants are independent of $(\mu,\theta)\in\mathscr{C}_\lambda$. To complete the proof, we observe that $c(\mu_\lambda,0)=\{(g\lambda/2\pi)\tanh(2\pi h/\lambda)\}^{\frac{1}{2}}\to0$ as $\lambda\to0$.

(vi) Since $c(\mu_n,\theta_n)\leqslant M$, it follows from (1.36) that $q_c(\mu_n,\theta_n)\to0$ as $n\to\infty$. The asymptotic behaviour of $\{\theta_n\}$ as $n\to\infty$ is established by a slight modification of the arguments in I, § 5, using (iv) to obtain the appropriate estimates. The behaviour of the limiting function θ may be analyzed by precisely the method used to establish the results in I, theorem 5.2 (*d*)–(*g*).

(vii) This is the main result of McLeod (1982) reformulated in terms of equation (2.1). The proof for equation (2.1) is identical (with certain obvious modifications), and there is no need to repeat it here. Since $\mathscr{C}_\lambda$ is a connected set in $\mathbb{R}\times C_0[0,\pi]$ which contains $(\mu_\lambda,0)$ and a point (μ,θ) with $\sup_{s\in[0,\pi]}\theta(s)\geqslant\frac{1}{6}\pi+\epsilon$, it is immediate that for each $\beta\in[0,\frac{1}{6}\pi+\epsilon]$ there exists an element $(\mu,\theta)\in\mathscr{C}_\lambda$ with $\sup_{s\in[0,\pi]}\theta(s)=\beta$.

(viii) We sketch the proof for $l=1$; for general l the result follows by induction. Let $(\mu,\theta)\in\mathscr{C}_\lambda$ with $\mu\leqslant N$. Then the odd extension of θ to $[-\pi,\pi]$ is the conjugate of the even function ρ defined in the proof of (ii) (for the L_2-definition of the conjugate operator, which is sufficient for our purposes here, see Appendix). Standard theory (Zygmund 1977) then gives that

$$|\theta|_{C^\alpha}\leqslant\text{const.}\times|\rho|_{C^\alpha},$$

where C^α denotes the Banach space of Hölder continuous functions on $[-\pi,\pi]$ with exponent $\alpha\in(0,1)$, and the constant depends only on α. For $s_1,s_2\in[-\pi,\pi]$,

$$\begin{aligned} |\rho(s_1)-\rho(s_2)| &= \tfrac{1}{3}\left|\ln\left[\frac{1+\mu\int_0^{s_1}f_\lambda\sin\theta}{1+\mu\int_0^{s_2}f_\lambda\sin\theta}\right]\right| \\ &\leqslant\text{const.}\times\tfrac{1}{3}\mu|s_1-s_2|\leqslant\text{const.}\times\tfrac{1}{3}N|s_1-s_2|. \end{aligned}$$

Thus $|\rho|_{C^\alpha}\leqslant$ const. and hence $|\theta|_{C^\alpha}\leqslant$ const., where the constant depends only on N, α and λ.

However, the function $\mathrm{d}\theta/\mathrm{d}s$ is the conjugate of

$$\frac{\mathrm{d}\rho}{\mathrm{d}s}=-\frac{1}{3}\frac{f_\lambda(s)\sin\theta(s)}{\dfrac{1}{\mu}+\int_0^s f_\lambda\sin\theta}$$

and so $|\mathrm{d}\rho/\mathrm{d}s|_{C^\alpha}\leqslant$ const., whence $|\mathrm{d}\theta/\mathrm{d}s|_{C^\alpha}\leqslant$ const., and once again the constant depends only upon N, α and λ. The result for $l=1$ then follows by the Ascoli–Arzela theorem. The proof in the general case follows once one has taken into account that the conjugate of the lth derivative of ρ is the lth derivative of θ.

Remark. The proof of theorem 2.2 (ii)–(vi), (viii) did not use the connectedness of $\mathscr{C}_\lambda$, and so all of these results hold with $\mathscr{C}_\lambda$ replaced by $\mathscr{S}_\lambda$.

2.3. *Properties of periodic waves*

In § 2.2 the global nature of the solution set of the periodic water-wave problem was studied through its formulation as the integral equation (1.31). This equation bears a striking resemblance to the approximation used in I, § 3.2 to prove the existence of large-amplitude solitary waves. In § 3 we shall adapt the proofs in I to prove that as $\lambda \to \infty$ the unbounded, closed connected sets $\mathscr{C}_\lambda$ converge, in a certain sense, to a global 'branch' of solutions of the solitary wave problem.

In this section, we exploit the integral equation (1.32) to gain further insight into the nature of periodic waves which lie on the bifurcating set $\mathscr{C}_\lambda$. These results do not seem immediately accessible from (1.31), and are new.

In this section, our interest is restricted to solutions (μ, Θ) of equation (1.32). Since Θ is an odd function on $[-\frac{1}{2}\lambda, \frac{1}{2}\lambda]$, it suffices to consider the equation

$$\Theta(\chi) = A(\mu, \Theta)(\chi) \equiv \frac{1}{3\pi}\int_0^{\frac{1}{2}\lambda} \ln\left|\frac{s_\lambda(2K_\lambda(\chi+\epsilon)/\lambda)}{s_\lambda(2K_\lambda(\chi-\epsilon)/\lambda)}\right| \frac{\sin\Theta(\epsilon)}{\dfrac{\Lambda}{\mu}+\displaystyle\int_0^\epsilon \sin\Theta(w)\,\mathrm{d}w}\,\mathrm{d}\epsilon, \qquad (2.8)$$

$\chi \in [0, \frac{1}{2}\lambda]$. Here Λ, s_λ and K_λ are defined in § 1.3; the positive parameters h and λ upon which they depend are chosen arbitrarily but are then *fixed*.

Since the domain $R_\lambda = \{(\chi, \eta): \chi \in (-\frac{1}{2}\lambda, \frac{1}{2}\lambda),\ \eta \in (-h, 0)\}$ is mapped conformally onto the cut unit disc $\mathscr{D}' = \{r\,\mathrm{e}^{\mathrm{i}t}: t \in (-\pi, \pi), r \in (0,1)\}$ by $\tilde{p}_\lambda$, the results of theorem 2.2 have implications for the solution set of (2.8), some of which are set out below. Let $\mathscr{T}_\lambda = \{(\mu, \Theta) \in (0, \infty) \times \mathscr{K}_0[0, \frac{1}{2}\lambda]: \Theta \neq 0$ and (μ, Θ) satisfies (2.8)$\} \cup \{(\mu_\lambda, 0)\}$, where $\mu_\lambda = 6\Lambda\pi\lambda^{-1}\coth(2\pi h/\lambda)$ is given in theorem 1.3. Where necessary, we shall identify $\Theta \in \mathscr{K}_0[0, \frac{1}{2}\lambda]$ with its odd extension to $[-\frac{1}{2}\lambda, \frac{1}{2}\lambda]$.

THEOREM 2.3. *Let $\mathscr{E}_\lambda$ denote the maximal connected subset of $\mathscr{T}_\lambda$ in $(0, \infty) \times C_0[0, \frac{1}{2}\lambda]$ which contains $(\mu_\lambda, 0)$. Then*

(i) $\mathscr{E}_\lambda = \{(\mu, \Theta): \Theta(\chi) = -\theta(p_\lambda(\chi)), \chi \in [0, \frac{1}{2}\lambda]$, *where* $(\mu, \theta) \in \mathscr{C}_\lambda\}$.

(ii) *$\mathscr{E}_\lambda$ is closed and unbounded.*

(iii) *If $\tilde{\Theta}$ denotes the harmonic function on R_λ with $\tilde{\Theta}(\chi + \mathrm{i}0) = \Theta(\chi)$, $\chi \in [-\frac{1}{2}\lambda, \frac{1}{2}\lambda]$, and $\tilde{\Theta} = 0$ elsewhere on ∂R_λ, then*

$$\left.\frac{\partial\tilde{\Theta}}{\partial\eta}\right|_{\chi+\mathrm{i}0} = \frac{1}{3}\frac{\sin\Theta(\chi)}{\dfrac{\Lambda}{\mu}+\displaystyle\int_0^\chi \sin\Theta(w)\,\mathrm{d}w}, \qquad (2.9)$$

$\chi \in [-\frac{1}{2}\lambda, \frac{1}{2}\lambda]$. *Furthermore, $\tilde{\Theta}$ is real-analytic on $\bar{R}_\lambda$; in particular, Θ is real-analytic on $[0, \frac{1}{2}\lambda]$. If Θ is non-trivial, then*

(iv) $\tilde{\Theta}_\chi(0, \eta) > 0, \tilde{\Theta}_\chi(\frac{1}{2}\lambda, \eta) < 0$ *for all* $\eta \in (-h, 0)$.

(v) $\tilde{\Theta}_\eta(\chi, \eta) > 0$ *for all* $(\chi, \eta) \in (0, \frac{1}{2}\lambda) \times [-h, 0]$.

(vi) $\tilde{\Theta}_{\eta\chi}(0, \eta) > 0$, $\tilde{\Theta}_{\eta\chi}(\frac{1}{2}\lambda, \eta) < 0$ *for all* $\eta \in (-h, 0)$.

Proof. Theorem 1.4 (i) and the maximality of $\mathscr{C}_\lambda$ and $\mathscr{E}_\lambda$ in $\mathscr{S}_\lambda$ and $\mathscr{T}_\lambda$, respectively, together prove (i). Parts (ii) and (iii) follow immediately from theorem 2.2. By theorem 2.2 (iii), $\tilde{\theta}$ is real-analytic on $\bar{\mathscr{D}}$, and hence $\tilde{\Theta}$ is real-analytic on $\bar{R}_\lambda$ since $\tilde{p}_\lambda$ is analytic there and $\tilde{\Theta}(\zeta) = -\tilde{\theta}(\tilde{p}_\lambda(\zeta))$. Equation (2.9) is a restatement of (1.33).

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

To prove (iv)–(vi), we use the maximum principle (Protter & Weinberger 1967). By the maximum principle for a harmonic function u on a rectangle R we mean the fact that $\min_{\partial R} u < u(\zeta) < \max_{\partial R} u$, for all $\zeta \in R$; while the strong maximum principle refers to the fact that at every point of ∂R, other than corners, where the maximum (minimum) of u is attained, the outward normal derivative is positive (negative). Let $R = (0, \frac{1}{2}\lambda) \times (-h, 0)$. Since $\tilde{\Theta}(\chi, 0) = \Theta(\chi) > 0$ on $(0, \frac{1}{2}\lambda)$ and vanishes elsewhere on ∂R, the strong maximum principle gives (iv) and the result $\tilde{\Theta}_\eta(\chi, -h) > 0, \chi \in (0, \frac{1}{2}\lambda)$. Since (2.9) ensures that $\tilde{\Theta}_\eta(\chi, 0) > 0$ for all $\chi \in (0, \frac{1}{2}\lambda)$, and since $\tilde{\Theta}_\eta$ vanishes on the lines $\{(\chi, \eta): \chi = 0, \frac{1}{2}\lambda, \eta \in [-h, 0]\}$, part (v) follows from the maximum principle. The strong maximum principle for $\tilde{\Theta}_\eta$ then gives (vi).

Theorem 2.3 (i) ensures that any properties proved for elements of $\mathscr{E}_\lambda$ may be translated into corresponding results for $\mathscr{C}_\lambda$. The next three theorems concern solutions of (2.8) with $0 < \Theta(\chi) \leqslant \pi$ on $(0, \frac{1}{2}\lambda)$, and note that the results hold for all elements of $\mathscr{E}_\lambda \backslash \{(\mu_\lambda, 0)\}$, since such elements satisfy $0 < \Theta(\chi) < \frac{1}{2}\pi$ on $(0, \frac{1}{2}\lambda)$ by theorem 2.2 (ii).

The following theorem ensures that non-trivial elements of $\mathscr{E}_\lambda$ satisfy $\chi\Theta'(\chi) < \Theta(\chi)$ on $(0, \frac{1}{2}\lambda]$, and, equivalently, that $\chi^{-1}\Theta(\chi)$ is monotone decreasing on $(0, \frac{1}{2}\lambda)$. This property implies that $\Theta(\chi) < \frac{1}{3}\pi, \chi \in [0, \frac{1}{2}\lambda]$, for all elements of $\mathscr{E}_\lambda$, and, equivalently, that $\theta(s) < \frac{1}{3}\pi$, $s \in [0, \pi]$, for all elements of $\mathscr{C}_\lambda$.

THEOREM 2.4.† *Assume that* (μ, Θ) *satisfies* (2.8) *and* $0 < \Theta(\chi) \leqslant \pi$ *on* $(0, \frac{1}{2}\lambda)$. *Then*

(i) $\chi\Theta'(\chi) < \Theta(\chi)$ *on* $(0, \frac{1}{2}\lambda]$

and

(ii) $0 < \Theta(\chi) < \frac{1}{3}\pi$ *on* $(0, \frac{1}{2}\lambda)$.

Proof. (i) Assume that (i) is false, and let $\tilde{\Theta}$ be as in theorem 2.3 (iii). Since $\tilde{\Theta}(\chi, -h) = 0$, there follows $\tilde{\Theta}_\chi(\chi, -h) = 0$, $\chi \in [0, \frac{1}{2}\lambda]$, and the use of this with theorem 2.3 (vi) ensures that

$$\tilde{\Theta}_\chi(0, 0) = \Theta'(0) > 0 \quad \text{and} \quad \tilde{\Theta}_\chi(\tfrac{1}{2}\lambda, 0) = \Theta'(\tfrac{1}{2}\lambda) < 0. \tag{2.10}$$

Hence there exists $\chi \in (0, \frac{1}{2}\lambda)$ such that $\chi\tilde{\Theta}_\chi(\chi, 0) \geqslant \tilde{\Theta}(\chi, 0)$. The use of this with (2.10) ensures that for some constant $d \geqslant 1$

$$\chi\tilde{\Theta}_\chi(\chi, 0) \leqslant d\tilde{\Theta}(\chi, 0) \quad \text{for all} \quad \chi \in [0, \tfrac{1}{2}\lambda], \tag{2.11a}$$

and

$$\hat{\chi}\tilde{\Theta}_\chi(\hat{\chi}, 0) = d\tilde{\Theta}(\hat{\chi}, 0) \quad \text{for some} \quad \hat{\chi} \in (0, \tfrac{1}{2}\lambda). \tag{2.11b}$$

Define a function W on $R = (0, \frac{1}{2}\lambda) \times (-h, 0)$ by

$$W(\chi, \eta) = \chi\tilde{\Theta}_\chi(\chi, \eta) - d\tilde{\Theta}(\chi, \eta).$$

It follows that W vanishes on the lines $\{(0, \eta): \eta \in (-h, 0)\}$ and $\{(\chi, -h): \chi \in (0, \frac{1}{2}\lambda)\}$; that W is negative on the line $\{(\frac{1}{2}\lambda, \eta): \eta \in (-h, 0)\}$ by theorem 2.3 (iv); and that W is non-positive on the line $\{(\chi, 0): \chi \in (0, \frac{1}{2}\lambda)\}$ by (2.11a). Hence $W \leqslant 0$ on ∂R. A calculation yields

$$\Delta W - \frac{2}{\chi} W_\chi - \frac{2}{\chi^2}(d-1)\, W = \frac{2d}{\chi^2}(d-1)\, \tilde{\Theta} \geqslant 0 \quad \text{in} \quad R. \tag{2.12}$$

† That this result might hold for periodic waves was suggested to us by J. B. McLeod, who attributed it to Professor T. B. Benjamin, F.R.S., in the case of solitary waves. Note that, in the periodic case, an even finer estimate may be established by the same method, namely

$$\sin\left(\frac{\pi\chi}{\lambda}\right) \Theta'(\chi) < \frac{\pi}{\lambda} \cos\left(\frac{\pi\chi}{\lambda}\right) \Theta(\chi), \quad \chi \in (0, \tfrac{1}{2}\lambda].$$

As $\lambda \to \infty$, this reduces to Benjamin's result for solitary waves.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Standard theory (Protter *et al.* 1967, pp. 64, 67) applied to (2.12) ensures that $W < 0$ in R and that the normal derivative of W is positive at any point on ∂R (other than the line $\chi = 0$ since (2.12) is singular there) where W equals zero. By (2.11 *b*), $W(\hat{\chi}, 0) = 0$ for some $\hat{\chi} \in (0, \frac{1}{2}\lambda)$, and so $W_\eta(\hat{\chi}, 0)$ must be positive. A calculation yields

$$0 < W_\eta(\hat{\chi}, 0) = \hat{\chi}\tilde{\Theta}_{\chi\eta}(\hat{\chi}, 0) - d\tilde{\Theta}_\eta(\hat{\chi}, 0)$$
$$= \hat{\chi}\frac{\mathrm{d}}{\mathrm{d}\chi}\left\{\frac{1}{3}\frac{\sin\Theta(\chi)}{\frac{\Lambda}{\mu}+\int_0^\chi \sin\Theta}\right\}\Bigg|_{\chi=\hat{\chi}} - \frac{d}{3}\left\{\frac{\sin\Theta(\chi)}{\frac{\Lambda}{\mu}+\int_0^\chi \sin\Theta}\right\}\Bigg|_{\chi=\hat{\chi}} < \frac{d}{3}\left\{\frac{\Theta(\hat{\chi})\cos\Theta(\hat{\chi}) - \sin\Theta(\hat{\chi})}{\frac{\Lambda}{\mu}+\int_0^{\hat{\chi}} \sin\Theta}\right\}, \tag{2.13}$$

where we have used the relation $\hat{\chi}\Theta'(\hat{\chi}) = d\Theta(\hat{\chi})$ from (2.11 *b*). Since $\Theta \leqslant \pi$, it follows that the right-hand side of (2.13) is negative, and this is the desired contradiction.

(ii) The arguments for theorem 2.2 (ii) show that $\Theta < \frac{1}{2}\pi$, and the use of this with (i) ensures that $\chi^{-1}\sin\Theta(\chi)$ is monotone decreasing on $(0, \frac{1}{2}\lambda)$. Hence,

$$\frac{\sin\Theta(\epsilon)}{\frac{\Lambda}{\mu}+\int_0^\epsilon \sin\Theta} < \frac{\sin\Theta(\epsilon)}{\int_0^\epsilon \sin\Theta} = \frac{\sin\Theta(\epsilon)}{\int_0^\epsilon \frac{\sin\Theta(w)}{w}\,w\,\mathrm{d}w} < \frac{\sin\Theta(\epsilon)}{\frac{\sin\Theta(\epsilon)}{\epsilon}\int_0^\epsilon w\,\mathrm{d}w} = \frac{2}{\epsilon} \quad \text{for all} \quad \epsilon \in (0, \tfrac{1}{2}\lambda).$$

The use of this estimate in (2.8) yields

$$\Theta(\chi) < \frac{2}{3\pi}\int_0^{\frac{1}{2}\lambda} \ln\left|\frac{s_\lambda(2K_\lambda(\chi+\epsilon)/\lambda)}{s_\lambda(2K_\lambda(\chi-\epsilon)/\lambda)}\right|\frac{\mathrm{d}\epsilon}{\epsilon} \quad \text{for all} \quad \chi \in (0, \tfrac{1}{2}\lambda),$$

and making the transformation $\epsilon = q_\lambda(t)$ and $\chi = q_\lambda(s)$, $s, t \in (-\pi, 0)$, gives in the notation of theorem 1.1,

$$\Theta(\chi) = -\theta(s) < -\frac{4}{3}\int_{-\pi}^0 G(s,t)\frac{q_\lambda'(t)}{q_\lambda(t)}\,\mathrm{d}t, \quad s \in (-\pi, 0).$$

Since θ is an odd function, there results that

$$\theta(s) < \frac{4}{3}\int_0^\pi G(s,t)\frac{q_\lambda'(t)}{q_\lambda(t)}\,\mathrm{d}t, \quad s \in (0, \pi). \tag{2.14}$$

Since $q_\lambda'(t) = -\Lambda f_\lambda(t)$, we have

$$\frac{q_\lambda'(t)}{q_\lambda(t)} = \frac{f_\lambda(t)}{\int_0^t f_\lambda(w)\,\mathrm{d}w}.$$

It is noted in the proof of lemma 3.2 that $f_\lambda(t)/f(t)$ is monotone decreasing on $(0, \pi)$, where $f(t) = \frac{1}{2}\sec\frac{1}{2}t$. It follows that

$$\frac{q_\lambda'(t)}{q_\lambda(t)} = \frac{f_\lambda(t)}{\int_0^t \frac{f_\lambda(w)}{f(w)}f(w)\,\mathrm{d}w} < \frac{f_\lambda(t)}{\frac{f_\lambda(t)}{f(t)}\int_0^t f(w)\,\mathrm{d}w} = \frac{f(t)}{\int_0^t f(w)\,\mathrm{d}w},$$

and so
$$\theta(s) < \frac{4}{3}\int_0^\pi G(s,t)\frac{\frac{1}{2}\sec\frac{1}{2}t}{\ln(\sec\frac{1}{2}t+\tan\frac{1}{2}t)}\,\mathrm{d}t, \quad s \in (0, \pi).$$

A simple calculation gives

$$\frac{\frac{1}{2}\sec\frac{1}{2}t}{\ln(\sec\frac{1}{2}t+\tan\frac{1}{2}t)} \leqslant \tfrac{1}{2}(\tan\tfrac{1}{2}t+\cot\tfrac{1}{2}t), \quad t \in (0, \pi),$$

and so
$$\theta(s) < \frac{2}{3}\int_0^\pi G(s,t)\,(\tan\tfrac{1}{2}t+\cot\tfrac{1}{2}t)\,\mathrm{d}t = \tfrac{1}{3}(\pi-t)+\tfrac{1}{3}t = \tfrac{1}{3}\pi.$$

The evaluation of the integral in the expression above is given by I, theorem 2.5 (*d*), (*e*).

Remark. A more precise estimate using the right-hand side of (2.14) is not possible since one can show that this quantity approaches $\frac{1}{3}\pi$ as $s \to 0$.

Obviously, part (ii) of theorem 2.3 follows from the abstract global bifurcation theory for positive operators (Dancer 1973) using the reproducing cone $\mathscr{K}_0[0, \frac{1}{2}\lambda]$ (see Keady *et al.* (1978)). The next results of this section (theorems 2.5 and 2.6) are a consequence of the observation that a smaller cone $\hat{\mathscr{K}}$ is more appropriate in the study of equation (2.8). Here $\hat{\mathscr{K}} = \{u \in \mathscr{K}_0[0, \frac{1}{2}\lambda]$: for all $\hat{\chi} \in [\frac{1}{4}\lambda, \frac{1}{2}\lambda]$ and for all $\chi^* \in [\frac{1}{2}\lambda - \hat{\chi}, \hat{\chi}]$, $u(\hat{\chi}) \leqslant u(\chi^*)\}$. Note that if $u \in \hat{\mathscr{K}}$, then u is non-increasing on $[\frac{1}{4}\lambda, \frac{1}{2}\lambda]$, and hence $\hat{\mathscr{K}}$ is not reproducing in $C_0[0, \frac{1}{2}\lambda]$. Our aim is to show that $\mathscr{E}_\lambda \subset (0, \infty) \times \hat{\mathscr{K}}$, and hence that $\Theta'(\chi) < 0$ on $[\frac{1}{4}\lambda, \frac{1}{2}\lambda]$ for all non-trivial $(\mu, \Theta) \in \mathscr{E}_\lambda$.

THEOREM 2.5. *If (μ, Θ) is as in theorem 2.4, then $\Theta \in \hat{\mathscr{K}}$.*

Proof. Let $\tilde{\Theta}$ be as in theorem 2.3 (iii). Let $\hat{\chi} \in (\frac{1}{4}\lambda, \frac{1}{2}\lambda]$ and $\chi^* \in (\frac{1}{2}\lambda - \hat{\chi}, \hat{\chi})$, and define $\hat{\alpha} \in (\frac{1}{4}\lambda, \frac{1}{2}\lambda)$ by $\hat{\alpha} = \frac{1}{2}(\hat{\chi} + \chi^*)$. To prove the theorem, we claim that it suffices to show that for all $\alpha \in (\frac{1}{4}\lambda, \frac{1}{2}\lambda)$

$$\tilde{\Theta}(\chi, 0) \geqslant \tilde{\Theta}(2\alpha - \chi, 0) \quad \text{for all} \quad \chi \in [2\alpha - \tfrac{1}{2}\lambda, \alpha]; \tag{2.15}$$

indeed, since $\chi^* \in [2\hat{\alpha} - \frac{1}{2}\lambda, \hat{\alpha}]$, it follows from (2.15) that $\Theta(\chi^*) = \tilde{\Theta}(\chi^*, 0) \geqslant \tilde{\Theta}(2\hat{\alpha} - \chi^*, 0) = \tilde{\Theta}(\hat{\chi}, 0) = \Theta(\hat{\chi})$.

Assume that (2.15) is false for some $\alpha \in (\frac{1}{4}\lambda, \frac{1}{2}\lambda)$. For each number $d \geqslant 1$, define the continuous function g by

$$g(d) = \min_{\chi \in [2\alpha - \frac{1}{2}\lambda, \alpha]} \{d\tilde{\Theta}(\chi, 0) - \tilde{\Theta}(2\alpha - \chi, 0)\}.$$

The function $\tilde{\Theta}(\chi, 0)$ is strictly positive on $[2\alpha - \frac{1}{2}\lambda, \alpha]$ since this closed interval is contained in $(0, \frac{1}{2}\lambda)$. Hence, $g(d)$ is positive for all sufficiently large d, and since $g(1) < 0$, there exists $D > 1$ such that $g(D) = 0$. It follows that

$$D\tilde{\Theta}(\chi, 0) \geqslant \tilde{\Theta}(2\alpha - \chi, 0) \quad \text{for all} \quad \chi \in [2\alpha - \tfrac{1}{2}\lambda, \alpha], \tag{2.16a}$$

and

$$D\tilde{\Theta}(\tilde{\chi}, 0) = \tilde{\Theta}(2\alpha - \tilde{\chi}, 0) \quad \text{for some} \quad \tilde{\chi} \in (2\alpha - \tfrac{1}{2}\lambda, \alpha). \tag{2.16b}$$

Let R^α denote the region $(2\alpha - \frac{1}{2}\lambda, \alpha) \times (-h, 0)$, and define a harmonic function V^α on R^α by

$$V^\alpha(\chi, \eta) = D\tilde{\Theta}(\chi, \eta) - \tilde{\Theta}(2\alpha - \chi, \eta)$$

for $(\chi, \eta) \in R^\alpha$. It follows with the use of (2.16*a*) that $V^\alpha \geqslant 0$ on ∂R^α, and the maximum principle then ensures that $V^\alpha > 0$ in R^α. Since $V^\alpha(\tilde{\chi}, 0) = 0$ by (2.16*b*), the strong maximum principle gives

$$\begin{aligned} 0 > V^\alpha_\eta(\tilde{\chi}, 0) &= D\tilde{\Theta}_\eta(\tilde{\chi}, 0) - \tilde{\Theta}_\eta(2\alpha - \tilde{\chi}, 0) \\ &= \frac{D \sin \Theta(\tilde{\chi})}{3\left(\dfrac{\Lambda}{\mu} + \displaystyle\int_0^{\tilde{\chi}} \sin \Theta\right)} - \frac{\sin \Theta(2\alpha - \tilde{\chi})}{3\left(\dfrac{\Lambda}{\mu} + \displaystyle\int_0^{2\alpha - \tilde{\chi}} \sin \Theta\right)} > \frac{D \sin \Theta(\tilde{\chi}) - \sin \Theta(2\alpha - \tilde{\chi})}{3\left(\dfrac{\Lambda}{\mu} + \displaystyle\int_0^{2\alpha - \tilde{\chi}} \sin \Theta\right)}. \end{aligned} \tag{2.17}$$

Equation (2.16*b*) yields

$$D \sin \Theta(\tilde{\chi}) = D \sin (\Theta(2\alpha - \tilde{\chi})/D) > \sin \Theta(2\alpha - \tilde{\chi}),$$

since $D > 1$ and $\Theta \leqslant \pi$ (indeed, $\Theta < \frac{1}{3}\pi$ by theorem 2.4 (i)). The use of this inequality in (2.17) yields a contradiction, and so we conclude that (2.15) holds.

The following theorem gives various properties of Θ implied by membership in $\hat{\mathscr{K}}$.

THEOREM 2.6. *Let* (μ, Θ) *be as in theorem* 2.4, *and let* $\tilde{\Theta}$ *be as in theorem* 2.3 (iii). *If, in addition,* $\hat{\chi} \in (\frac{1}{4}\lambda, \frac{1}{2}\lambda)$ *and* $\chi^* \in [\frac{1}{2}\lambda - \hat{\chi}, \hat{\chi})$, *then*

(i) $\tilde{\Theta}(\hat{\chi}, \eta) < \tilde{\Theta}(\chi^*, \eta)$ *for all* $\eta \in (-h, 0]$; *and, in particular,* $\Theta(\hat{\chi}) < \Theta(\chi^*)$.

(ii) $\tilde{\Theta}_\eta(\hat{\chi}, \eta) < \tilde{\Theta}_\eta(\chi^*, \eta)$ *for all* $\eta \in [-h, 0]$. *Moreover,*

(iii) $\tilde{\Theta}_\chi(\chi, \eta) < 0$ *and* $\tilde{\Theta}_{\chi\eta}(\chi, \eta) < 0$ *for all* $(\chi, \eta) \in [\frac{1}{4}\lambda, \frac{1}{2}\lambda] \times (-h, 0]$; *in particular,* $\Theta'(\chi) < 0$ *for* $\chi \in [\frac{1}{4}\lambda, \frac{1}{2}\lambda]$.

(iv) $\tilde{\Theta}_\chi(0, \eta) + \tilde{\Theta}_\chi(\frac{1}{2}\lambda, \eta) > 0$, *and* $\tilde{\Theta}_{\chi\eta}(0, \eta) + \tilde{\Theta}_{\chi\eta}(\frac{1}{2}\lambda, \eta) > 0$ *for all* $\eta \in (-h, 0]$; *and, in particular,* $\Theta'(0) + \Theta'(\frac{1}{2}\lambda) > 0$.

Proof. (i) Since $\Theta \neq 0$, we know from theorem 2.4 (i) that $0 < \Theta(\chi) < \frac{1}{3}\pi$ on $(0, \frac{1}{2}\lambda)$. Combining this with the fact that $\Theta \in \hat{\mathscr{K}}$, yields that for $\chi_1 \in (\frac{1}{4}\lambda, \frac{1}{2}\lambda]$ and $\chi_2 \in [\frac{1}{2}\lambda - \chi_1, \chi_1)$,

$$\frac{\sin \Theta(\chi_1)}{\Lambda/\mu + \int_0^{\chi_1} \sin \Theta} < \frac{\sin \Theta(\chi_2)}{\Lambda/\mu + \int_0^{\chi_2} \sin \Theta}. \tag{2.18}$$

Now suppose $\hat{\chi} \in (\frac{1}{4}\lambda, \frac{1}{2}\lambda)$ and $\chi^* \in [\frac{1}{2}\lambda - \hat{\chi}, \hat{\chi})$, and put $\alpha = \frac{1}{2}(\chi^* + \hat{\chi})$. Define a harmonic function W^α on R^α by putting

$$W^\alpha(\chi, \eta) = \tilde{\Theta}(\chi, \eta) - \tilde{\Theta}(2\alpha - \chi, \eta)$$

for all $(\chi, \eta) \in R^\alpha = (2\alpha - \frac{1}{2}\lambda, \alpha) \times (-h, 0)$. Then

$$W^\alpha(2\alpha - \tfrac{1}{2}\lambda, \eta) > 0, \quad \eta \in (-h, 0];$$

$$W^\alpha_\eta(\chi, 0) > 0, \quad \chi \in (2\alpha - \tfrac{1}{2}\lambda, \alpha),$$

by (2.9) and (2.18); and $W^\alpha = 0$ elsewhere on ∂R^α. By the maximum principle $W^\alpha > 0$ on R^α, and by the strong maximum principle $W^\alpha(\chi, 0) > 0$ for all $\chi \in (2\alpha - \frac{1}{2}\lambda, \alpha)$. In particular, for $\chi = \chi^* \in (2\alpha - \frac{1}{2}\lambda, \alpha)$, there results that

$$\tilde{\Theta}(\chi^*, \eta) - \tilde{\Theta}(\hat{\chi}, \eta) > 0$$

for all $\eta \in (-h, 0]$, and (i) has been established.

(ii) We first prove (ii) for $\eta = -h$ and $\eta = 0$. Since $W^\alpha > 0$ on R^α and is zero on the line $\{(\chi, -h): \chi \in [2\alpha - \frac{1}{2}\lambda, \alpha]\}$, the strong maximum principle for W^α gives $W^\alpha_\eta(\chi, -h) > 0$, $\chi \in (2\alpha - \frac{1}{2}\lambda, \alpha)$. If we set $\chi = \chi^* \in (2\alpha - \frac{1}{2}\lambda, \alpha)$, then the case $\eta = -h$ is proved. It was shown in the proof of (i) that $W^\alpha_\eta(\chi, 0) > 0$, $\chi \in (2\alpha - \frac{1}{2}\lambda, \alpha)$, and so the result for $\eta = 0$ follows upon setting $\chi = \chi^*$.

We now show that $W^\alpha_\eta(\chi, \eta) > 0$ on R^α, so that the result for $\eta \in (-h, 0)$ in (ii) will follow upon setting $\chi = \chi^*$. Because of the maximum principle for W^α_η, it suffices to show that $W^\alpha_\eta \geqslant 0$ on ∂R^α; note that this has already been done for the horizontal portions of the boundary. For $\eta \in (-h, 0)$, we have

$$W^\alpha_\eta(\alpha, \eta) = \tilde{\Theta}_\eta(\alpha, \eta) - \tilde{\Theta}_\eta(\alpha, \eta) = 0,$$

and

$$W^\alpha_\eta(2\alpha - \tfrac{1}{2}\lambda, \eta) = \tilde{\Theta}_\eta(2\alpha - \tfrac{1}{2}\lambda, \eta) - \tilde{\Theta}_\eta(\tfrac{1}{2}\lambda, \eta) = \tilde{\Theta}_\eta(2\alpha - \tfrac{1}{2}\lambda, \eta) \geqslant 0$$

by theorem 2.3 (v).

(iii) If $\alpha \in [\frac{1}{4}\lambda, \frac{1}{2}\lambda)$, and W^α is the harmonic function defined on the region R^α as above, then it follows by the strong maximum principle that

$$W^\alpha_\chi(\alpha, \eta) < 0, \tag{2.19}$$

for all $\eta \in (-h, 0)$, whence, putting $\alpha = \chi$,

$$\tilde{\Theta}_\chi(\chi, \eta) + \tilde{\Theta}_\chi(\chi, \eta) < 0, \quad (\chi, \eta) \in [\tfrac{1}{4}\lambda, \tfrac{1}{2}\lambda) \times (-h, 0). \tag{2.20}$$

(Although (2.19) only proves (2.20) for $\chi < \frac{1}{2}\lambda$, the result for $\chi = \frac{1}{2}\lambda$ is due to theorem 2.3 (iv).) Differentiating (2.9) with respect to χ yields that

$$3\tilde{\Theta}_{\chi\eta}|_{(\chi,0)} = \frac{\Theta_\chi \cos\Theta}{\frac{\Lambda}{\mu}+\int_0^\chi \sin\Theta} - \frac{\sin^2\Theta}{\left(\frac{\Lambda}{\mu}+\int_0^\chi \sin\Theta\right)^2}, \tag{2.21}$$

which, combined with (i) above, yields

$$\tilde{\Theta}_{\chi\eta}|_{(\chi,0)} < 0, \quad \chi\in[\tfrac{1}{4}\lambda, \tfrac{1}{2}\lambda). \tag{2.22}$$

Hence, $\tilde{\Theta}_\chi$ does not attain its maximum on the line segment $\{(\chi,0): \chi\in(\frac{1}{4}\lambda,\frac{1}{2}\lambda)\}$, by the strong maximum principle. Combining (2.20), (2.22) and theorem 2.3 (vi) yields that $\tilde{\Theta}_\chi(\frac{1}{4}\lambda,0)$, $\tilde{\Theta}_\chi(\frac{1}{2}\lambda,0) < 0$, and the first part of (iii) has been established.

We now prove the second part of (iii). Since $\tilde{\Theta}_\chi(\frac{1}{2}\lambda,0)<0$, equation (2.21) ensures that $\tilde{\Theta}_{\chi\eta}(\frac{1}{2}\lambda,0)<0$, and the use of this with (2.22) proves the case $\eta=0$. It was shown in the proof of (ii) that $W^\alpha_\eta > 0$ on R^α and that $W^\alpha_\eta(\alpha,\eta)=0$, $\eta\in(-h,0)$, for all $\alpha\in[\frac{1}{4}\lambda,\frac{1}{2}\lambda)$. By the strong maximum principle,

$$0 > W^\alpha_{\eta\chi}(\alpha,\eta) = \tilde{\Theta}_{\eta\chi}(\alpha,\eta)+\tilde{\Theta}_{\eta\chi}(\alpha,\eta),$$

whence $\tilde{\Theta}_{\eta\chi}(\chi,\eta)<0$ for all $(\chi,\eta)\in[\frac{1}{4}\lambda,\frac{1}{2}\lambda)\times(-h,0)$. This, along with theorem 2.3 (vi) establishes (iii).

(iv) The function $W^{\frac{1}{4}\lambda}$ is a positive, harmonic function on $R^{\frac{1}{4}\lambda}$, and is zero on the line $\{(0,\eta): \eta\in(-h,0)\}$. Hence, by the strong maximum principle,

$$W^{\frac{1}{4}\lambda}_\chi(0,\eta) > 0, \quad \eta\in(-h,0).$$

Therefore
$$\tilde{\Theta}_\chi(0,\eta)+\tilde{\Theta}_\chi(\tfrac{1}{2}\lambda,\eta) > 0, \quad \eta\in(-h,0),$$

whence, by (iii),
$$\tilde{\Theta}_\chi(0,0) \geqslant -\tilde{\Theta}_\chi(\tfrac{1}{2}\lambda,0) > 0. \tag{2.23}$$

However, by (2.21),

$$\tilde{\Theta}_{\chi\eta}(0,0)+\tilde{\Theta}_{\chi\eta}(\tfrac{1}{2}\lambda,0) = \frac{\tilde{\Theta}_\chi(0,0)}{\frac{\Lambda}{\mu}} + \frac{\tilde{\Theta}_\chi(\frac{1}{2}\lambda,0)}{\frac{\Lambda}{\mu}+\int_0^{\frac{1}{2}\lambda}\sin\Theta} > 0, \quad \text{by (2.23)}, \tag{2.24}$$

and the first part of (iv) has been established.

It was shown in the proof of (ii) that $W^\alpha_\eta > 0$ on R^α for all $\alpha\in[\frac{1}{4}\lambda,\frac{1}{2}\lambda)$. For the case $\alpha=\frac{1}{4}\lambda$, we have $W^{\frac{1}{4}\lambda}_\eta = 0$ on the lines $\{(\chi,\eta): \chi=0, \frac{1}{4}\lambda, \eta\in(-h,0)\}$. By the strong maximum principle,

$$0 < W^{\frac{1}{4}\lambda}_{\chi\eta}(0,\eta) = \tilde{\Theta}_{\chi\eta}(0,\eta)+\tilde{\Theta}_{\chi\eta}(\tfrac{1}{2}\lambda,\eta), \quad \eta\in(-h,0),$$

which, together with (2.24), yields

$$\tilde{\Theta}_{\chi\eta}(0,\eta)+\tilde{\Theta}_{\chi\eta}(\tfrac{1}{2}\lambda,\eta) > 0, \quad \eta\in(-h,0].$$

Our aim at the outset was to design a cone which was invariant under the operator in equation (2.8), and which was sufficiently sophisticated in its structure to give information about the shape of solutions Θ, at least for large μ.

Our motivation comes from various numerical results (Cokelet 1977, p. 215; Schwartz 1974, p. 572; Thomas 1968, pp. 146–147) which make it seem plausible that Θ' should have a unique zero in $(0,\frac{1}{2}\lambda)$, if (μ,Θ) is in $\mathscr{E}_\lambda$. Physically, all this says is that the wave has only one inflection point between crest and trough; theorem 2.6 says that there are none between $\frac{1}{4}\lambda$ and $\frac{1}{2}\lambda$, but this does not appear to help. The Serrin–Lavrentiev comparison theorems have been suggested as a possible way to tackle the problem (Keady & Pritchard 1974, pp. 365–367) but there are difficulties in applying them in this case (Toland 1978, p. 484). (However there are other indications (McLeod 1982) which suggest that the number of zeros of Θ' approaches infinity as $\mu \to \infty$.)

From the point of view of this section, a natural approach is to let $\mathcal{N}$ be the set of all solutions in $\mathscr{E}_\lambda$ which have the property that Θ' vanishes only once in $(0, \frac{1}{2}\lambda)$. One can use the local bifurcation theory to show that $\mathcal{N}$ is not empty and non-trivial, and it is clearly closed. We have been unable to show it to be open, but remark that it suffices to show that Θ' and Θ'' cannot vanish simultaneously on $[0, \frac{1}{2}\lambda]$. For the analogous problem in the theory of nonlinear Sturm–Liouville problems (Rabinowitz 1971, pp. 500–503) this method works, since there Θ' and Θ'' cannot vanish simultaneously (because of the uniqueness theorem for differential equations).

Finally, we remark that numerical evidence suggests that the zeros of Θ' approach 0 as μ approaches infinity; Θ' being negative on $(0, \frac{1}{2}\lambda)$ in the limiting case of $1/\mu = 0$, which means that the limiting wave is convex (Schwartz 1974, p. 576; Thomas 1968, p. 147).

2.4. *Firm conclusions about the periodic water-wave problem – a summary*

Roughly speaking, the following firm conclusions have been reached about the periodic water-waves under consideration (i.e. those which correspond to solutions in $\mathscr{C}_\lambda$ of Nekrasov's integral equation). The free surface Γ_λ is the graph of a real-analytic function of period λ provided the solution (μ, θ) of Nekrasov's equation to which it corresponds has $\mu < \infty$. Moreover in this case the slope of the wave profile never exceeds $\sqrt{3}$, but there do exist waves whose slope exceeds $1/\sqrt{3}$. The mean velocity of all waves is bounded above by an absolute constant which is independent of λ, and is bounded below by a constant which depends on λ and which tends to 0 as $\lambda \to 0$. Apart from the *a priori* bound of $\sqrt{3}$ for the wave slope, the main conclusion of § 2.3 is that the profile of a periodic wave is convex in the half wavelength centred about each trough.

The unboundedness of $\mathscr{C}_\lambda$ ensures the existence of a sequence (μ_n, θ_n) in $\mathscr{C}_\lambda$ with $\mu_n \to \infty$ as $n \to \infty$. This means that the speed of the flow at the crest of the corresponding waves tends to 0 as $n \to \infty$. We know that there exists a subsequence of these waves for which the profiles tend to the profile of a periodic wave which has zero speed at its crests (i.e. it has a stagnation point there). This wave corresponds to a solution of Nekrasov's equation with $\mu = \infty$, and its profile is the graph of a real-analytic function except at its crests. (In the light of the footnote on page 649, it is now known that the first derivative of the function describing the profile has a simple jump discontinuity at each crest.)

3. On the convergence of periodic waves to solitary waves in the long-wave limit

Throughout this section the mean depth h is *fixed*. The purpose here is to show the sense in which the sets $\mathscr{C}_\lambda$ of periodic water-waves converge to a set $\mathscr{C}'$ of solitary waves as the wavelength increases indefinitely. Recall from § 1.2, that each set $\mathscr{C}_\lambda$ contains exactly one point corresponding to a uniform horizontal flow of depth h, and that this point $(\mu_\lambda, 0)$ is the point at which periodic waves of wavelength λ and mean depth h bifurcate. In other words, on a flow of depth h, periodic waves of wavelength λ bifurcate from the horizontal flow when the mean velocity of the flow is $\{(g\lambda/2\pi) \tanh (2\pi h/\lambda)\}^{\frac{1}{2}}$. Moreover, the value of μ_λ converges to $6/\pi$ as $\lambda \to \infty$ (theorem 1.3).

Let U be any bounded, open set in $\mathbb{R} \times C_0[0, \pi]$ such that $(6/\pi, 0) \in U$. Then, for all λ sufficiently large, $\mathscr{C}_\lambda \cap \partial U \neq \varnothing$. The next theorem is the main result of this paper. (Further properties of the function θ constructed below are given in the remarks following theorem 3.5; in particular, part (i) may be improved to $0 < \theta(s) < \frac{1}{3}\pi$ on $(0, \pi)$.)

THEOREM 3.1. *Suppose* $\{\lambda_n\} \subset \mathbb{R}$ *and* $\lambda_n \uparrow \infty$ *as* $n \to \infty$, *and suppose that* $\mathscr{C}_{\lambda_n} \cap \partial U \neq \varnothing$ *for each n. If* $\{(\mu_n, \theta_n)\} \subset (0, \infty) \times \mathscr{K}_0$ *is a sequence such that* $(\mu_n, \theta_n) \in \mathscr{C}_{\lambda_n} \cap \partial U$ *for each n, then the sequence* $\{(\mu_n, \theta_n)\}$ *is relatively compact in* $[6/\pi, \infty) \times \mathscr{K}_0$. *If* $\{(\mu_{n(k)}, \theta_{n(k)})\}$ *is a subsequence of* $\{(\mu_n, \theta_n)\}$ *such that*

$$(\mu_{n(k)}, \theta_{n(k)}) \to (\mu, \theta) \in [6/\pi, \infty) \times \mathscr{K}_0, \tag{3.1}$$

then

(i) $\mu > 6/\pi$, $0 < \theta(s) < \frac{1}{2}\pi$ *on* $(0, \pi)$ *and* $(\mu, \theta) \in \partial U$.

(ii) (μ, θ) *is a solution of the equation for solitary waves* (1.47).

(iii) *The sequence* $\{f_{\lambda_{n(k)}}, \theta_{n(k)}\}$ *converges in* $L_1(0, \pi)$ *to* $f\theta$ *as* $k \to \infty$.

(iv) *If* $c(\mu_{n(k)}, \theta_{n(k)})$ *is calculated using* $\lambda_{n(k)}$ *instead of* λ *in expression* (1.34), *then*

$$c(\mu_{n(k)}, \theta_{n(k)}) \to \left\{\frac{6gh}{\pi}\left(\frac{1}{\mu} + \int_0^\pi f(t) \sin\theta(t)\, \mathrm{d}t\right)\right\}^{\frac{1}{2}} \in (\surd(gh), 2\surd(gh)).$$

(v) *For each k, the free surface may be denoted by* $\{(x, H_k(x)) : x \in (-\frac{1}{2}\lambda_{n(k)}, \frac{1}{2}\lambda_{n(k)})\}$ *where* H_k *depends on* $\lambda_{n(k)}$, $\mu_{n(k)}$ *and* $\theta_{n(k)}$ *according to the formulae* (1.37) *and* (1.38). *As* $k \to \infty$,

$$H_k(x) - H_k(0) \to H(x) - H(0),$$

uniformly on compact intervals, where $\{(x, H(x)) : x \in \mathbb{R}\}$ *is the profile of the solitary wave corresponding to the solution* (μ, θ) *of* (1.47). *The function H may be calculated from* (μ, θ) *by the formulae* (1.48), (1.49).

A proof of this theorem may be obtained by modifying the arguments of I, theorem 3.8. The following lemmas facilitate this procedure.

LEMMA 3.2. *For any non-negative, bounded function u on* $[0, \pi]$, *whose support has full measure, and for any* $\alpha \geqslant 0$,

$$\frac{f_\lambda(t)\, u(t)}{\alpha + \int_0^t f_\lambda(w)\, u(w)\, \mathrm{d}w} \geqslant \frac{f_\nu(t)\, u(t)}{\alpha + \int_0^t f_\nu(w)\, u(w)\, \mathrm{d}w},$$

if $\lambda \geqslant \nu > 0$, *and* f_λ, f_ν *are defined by the expression* (1.22).

Proof. Since $f_\lambda(t) \geqslant f_\nu(t)$ for all $t \in [0, \pi]$ when $\lambda \geqslant \nu$, it will suffice to show that

$$f_\lambda(t) \int_0^t f_\nu(w)\, u(w)\, \mathrm{d}w \geqslant f_\nu(t) \int_0^t f_\lambda(w)\, u(w)\, \mathrm{d}w$$

for all $t \in [0, \pi]$. In other words, it will suffice to show that

$$\begin{aligned} 0 &\leqslant \int_0^t (f_\lambda(t) f_\nu(w) - f_\nu(t) f_\lambda(w))\, u(w)\, \mathrm{d}w \\ &= \int_0^t f_\nu(t) f_\nu(w) \left(\frac{f_\lambda(t)}{f_\nu(t)} - \frac{f_\lambda(w)}{f_\nu(w)}\right) u(w)\, \mathrm{d}w. \end{aligned}$$

However, a simple calculation yields that f_λ/f_ν is increasing on $(0, \pi)$, and the proof is complete.

LEMMA 3.3. *For each* $\lambda > 1/\pi$, *let* g_λ *denote the function defined on* $[0, \pi]$ *by putting*

$$g_\lambda(s) = \begin{cases} f_\lambda(s), s \in [0, \pi - 1/\lambda], \\ 0, s \in (\pi - 1/\lambda, \pi]. \end{cases} \tag{3.2}$$

Then there exists a unique solution $(\gamma_\lambda, \psi_\lambda)$ *of*

$$\psi(s) = \frac{2\gamma}{3}\int_0^\pi G(s,t)\, g_\lambda(t)\, \psi(t)\, \mathrm{d}t,$$

with $(\gamma, \psi) \in [0, \infty) \times \mathscr{K}_0$ *and* $|\psi|_{C_0[0,\pi]} = 1$. *Moreover* $\gamma_\lambda \downarrow 6/\pi$ *as* $\lambda \to \infty$.

Proof. The proof is similar to that of I, theorem 3.2. Existence and uniqueness follow immediately from the general theory of u_0-positive linear operators, and that $\gamma_\lambda \downarrow 6/\pi$ follows by exactly the same argument as was used to show that $\gamma_n \downarrow 6/\pi$ in I, theorem 3.2.

The proof of theorem 3.1 consists of a number of steps. Since $\mathscr{C}_\lambda \cap \partial U \neq \varnothing$ we can find certain elements of this intersection which converge (in a sense to be made precise) to an element (μ, θ) as $\lambda \to \infty$. We then show that θ is non-trivial, and that various physical quantities such as the mean velocity converge as $\lambda \to \infty$. Then, as a consequence of theorem 1.5, the convergence of the wave profile to that of a solitary wave may be inferred.

Proof of theorem 3.1. Because of the obvious similarity between the problem here and that of proving theorem 3.9 of I we shall limit ourselves to giving an outline of the proof. The letters (A'), (B'), (C'), etc., when used below, refer to the points in the proof of theorem 3.9 of I so labelled.

Since $\{(\mu_n, \theta_n)\} \subset \partial U \subset \mathbb{R} \times C_0[0, \pi]$ is a bounded sequence, there exists a subsequence $\{(\mu_{n(k)}, \theta_{n(k)})\}$ and a corresponding sequence $\{\lambda_{n(k)}\} \subset \mathbb{R}$, such that

$$\mu_{n(k)} \to \mu \quad \text{in} \quad \mathbb{R}, \tag{3.3a}$$

$$\theta_{n(k)} \rightharpoonup \theta \quad \text{weakly in} \quad L_2(0, \pi), \tag{3.3b}$$

$$\sin\theta_{n(k)} \rightharpoonup \sigma \quad \text{weakly in} \quad L_2(0, \pi) \tag{3.3c}$$

and

$$\lambda_{n(k)} \uparrow \infty \quad \text{in} \quad \mathbb{R}$$

as $k \to \infty$. We shall show that the conclusions (i)–(v) of the theorem hold for this subsequence. For the sake of having a convenient notation, we shall henceforth use $\{\mu_n\}$, $\{\theta_n\}$, $\{\lambda_n\}$ to denote the subsequence for which (3.3) holds.

(i), (ii), (iii) An obvious adaptation of (A′)–(D′) yields that $\theta_n \to \theta$ and $\sin\theta_n \to \sin\theta$ in $L_2(0, \pi)$ as $n \to \infty$; that $\theta_n \to \theta$ in $C[0, \delta]$ for each $\delta \in (0, \pi)$; and that $(\mu, \theta) \in [6/\pi, \infty) \times \mathscr{K}_0$ is a solution of (1.47). The next step is to prove that θ is non-trivial. To do this we first show that if $\theta = 0$, then $\mu = 6/\pi$.

Now for each n,

$$\theta_n(s) = \frac{2}{3}\int_0^\pi G(s,t)\, \frac{f_{\lambda_n}(t)\sin\theta_n(t)}{\dfrac{1}{\mu_n} + \displaystyle\int_0^t f_{\lambda_n}(w)\sin\theta_n(w)\,\mathrm{d}w}\,\mathrm{d}t$$

$$\geqslant \frac{2}{3}\int_0^\pi G(s,t)\, \frac{f_{\lambda_l}(t)\sin\theta_n(t)}{\dfrac{1}{\mu_n} + \displaystyle\int_0^t f_{\lambda_l}(w)\sin\theta_n(w)\,\mathrm{d}w}$$

for all $n \geqslant l$ (by lemma 3.2 and the fact that $\lambda_n \uparrow \infty$),

$$\geqslant \frac{2}{3}\int_0^\pi G(s,t)\, \frac{g_{\lambda_l}(t)\sin\theta_n(t)}{\dfrac{1}{\mu_n} + \displaystyle\int_0^t f_{\lambda_l}(w)\sin\theta_n(w)\,\mathrm{d}w}\,\mathrm{d}t,$$

where g_{λ_l} is defined by (3.2). Therefore

$$\theta_n(s) \geqslant \tfrac{2}{3}A_{n,l}\left\{\frac{\int_0^\pi G(s,t)\,g_{\lambda_l}(t)\,\theta_n(t)\,\mathrm{d}t}{\dfrac{1}{\mu_n}+\int_0^\pi f_{\lambda_l}(w)\sin\theta_n(w)\,\mathrm{d}w}\right\}, \tag{3.4}$$

for all $s\in[0,\pi]$, where
$$A_{n,l} = \inf_{s\in[0,\pi-1/\lambda_l]} \frac{\sin\theta_n(s)}{\theta_n(s)}.$$

Now multiplying this inequality by $g_{\lambda_l}\psi_{\lambda_l}$, whose existence is guaranteed by lemma 3.3, and integrating gives

$$\gamma_{\lambda_l}\int_0^\pi \theta_n(s)\,\psi_{\lambda_l}(s)\,g_{\lambda_l}(s)\,\mathrm{d}s \geqslant A_{n,l}\left\{\frac{\int_0^\pi g_{\lambda_l}(t)\,\psi_{\lambda_l}(t)\,\theta_n(t)\,\mathrm{d}t}{\dfrac{1}{\mu_n}+\int_0^\pi f_{\lambda_l}(w)\sin\theta_n(w)\,\mathrm{d}w}\right\}.$$

Thus
$$\frac{1}{\mu_n}+\int_0^\pi f_{\lambda_l}(w)\sin\theta_n(w)\,\mathrm{d}w \geqslant A_{n,l}/\gamma_{\lambda_l}$$

for all $n \geqslant l$. If $\theta_n \to 0$ in $L_2(0,\pi)$ as $n\to\infty$, then $\theta_n \to 0$ in $C[0,\delta]$ for each $\delta\in(0,\pi)$, and so $A_{n,l}\to 1$ as $n\to\infty$ for each fixed l. There results that

$$1/\mu = \lim_{n\to\infty} 1/\mu_n \geqslant \gamma_{\lambda_l}^{-1}$$

for each l. Since $\gamma_{\lambda_l}\downarrow 6/\pi$, as $l\uparrow\infty$, it follows that $\mu\leqslant 6/\pi$. Since $\mu_n > \mu_{\lambda_n}\downarrow 6/\pi$ as $n\to\infty$ (theorems 2.2 (ii) and 1.3) it follows that $\mu\geqslant 6/\pi$. We have shown that if $(\mu_n,\theta_n)\to(\mu,0)$ in $\mathbb{R}\times L_2(0,\pi)$, then $\mu = 6/\pi$.

From this observation, the method of (F') yields that $f_{\lambda_n}\theta_n$ converges to $f\theta$ in $L_1(0,\pi)$, and then the method of (G') may be used to prove that $\theta_n\to\theta$ in $C_0[0,\pi]$. The function θ must therefore be non-zero, for otherwise, as we have seen, $(\mu_n,\theta_n)\to(6/\pi,0)$ in $\mathbb{R}\times C_0[0,\pi]$. This contradicts the fact that U is an open set which contains $(6/\pi,0)$ in its interior. That $0<\theta(s)<\frac{1}{2}\pi$ on $(0,\pi)$, and $\mu > 6/\pi$ is proved in I, theorem 3.7. This completes the proof of (i), (ii), (iii).

(iv) By (1.23) and (1.34)

$$c(\mu_n,\theta_n) = \frac{2K_{\lambda_n}(1+k_{\lambda_n})\sqrt{(3g)}}{\lambda_n}\left(\frac{2}{\lambda_n}\int_0^\pi \frac{f_{\lambda_n}(t)\cos\theta_n(t)}{\left(\dfrac{1}{\mu_n}+\int_0^t f_{\lambda_n}(w)\sin\theta_n(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}t\right)^{-\frac{3}{2}}.$$

From (1.24) it follows that
$$\frac{2K_{\lambda_n}(1+k_{\lambda_n})\sqrt{(3g)}}{\lambda_n} \to \frac{\pi\sqrt{(3g)}}{2h} \tag{3.5}$$

as $n\to\infty$. Now for any $\alpha\in(0,\pi)$,

$$\frac{2}{\lambda_n}\int_0^\pi \frac{f_{\lambda_n}(t)\cos\theta_n(t)}{\left(\dfrac{1}{\mu_n}+\int_0^t f_{\lambda_n}(w)\sin\theta_n(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}t$$
$$= \frac{2}{\lambda_n}\int_0^\alpha \frac{f_{\lambda_n}(t)\cos\theta_n(t)}{\left(\dfrac{1}{\mu_n}+\int_0^t f_{\lambda_n}(w)\sin\theta_n(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}t + \frac{2}{\lambda_n}\int_\alpha^\pi \frac{f_{\lambda_n}(t)\cos\theta_n(t)}{\left(\dfrac{1}{\mu_n}+\int_0^t f_{\lambda_n}(w)\sin\theta_n(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}t. \tag{3.6}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

For any $\epsilon > 0$, choose $\alpha(\epsilon) \in (0, \pi)$ such that for all n sufficiently large $|\cos\theta_n(t) - 1| \leqslant \epsilon$ for all $t \in [\alpha(\epsilon), \pi]$. This can be done since $\theta_n \to \theta \in \mathscr{K}_0$ uniformly on $[0, \pi]$. Moreover, by (3.3a) and (iii), $\alpha(\epsilon)$ can be chosen so that

$$\left|\left(\frac{1}{\mu_n} + \int_0^t f_{\lambda_n}(w)\sin\theta_n(w)\,\mathrm{d}w\right)^{\frac{1}{3}} - \left(\frac{1}{\mu} + \int_0^\pi f(w)\sin\theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}\right| \leqslant \epsilon$$

for all $t \in [\alpha(\epsilon), \pi]$ and for all n sufficiently large. From (3.6) it follows that

$$\lim_{n\to\infty}\frac{2}{\lambda_n}\int_0^\pi \frac{f_{\lambda_n}(t)\cos\theta_n(t)}{\left(\frac{1}{\mu_n} + \int_0^t f_{\lambda_n}(w)\sin\theta_n(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\mathrm{d}t = \lim_{n\to\infty}\frac{2}{\lambda_n}\int_{\alpha(\epsilon)}^\pi \frac{f_{\lambda_n}(t)\cos\theta_n(t)}{\left(\frac{1}{\mu_n} + \int_0^t f_{\lambda_n}(w)\sin\theta_n\,\mathrm{d}w\right)^{\frac{1}{3}}}\mathrm{d}t$$

$$\begin{cases} \geqslant \lim\limits_{n\to\infty}\dfrac{2}{\lambda_n}\displaystyle\int_{\alpha(\epsilon)}^\pi \dfrac{f_{\lambda_n}(t)\,(1-\epsilon)}{\left(\dfrac{1}{\mu_n} + \displaystyle\int_0^\pi f_{\lambda_n}(w)\sin\theta_n(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\mathrm{d}t \\ \leqslant \lim\limits_{n\to\infty}\dfrac{2}{\lambda_n}\displaystyle\int_{\alpha(\epsilon)}^\pi \dfrac{f_{\lambda_n}(t)}{\left(\dfrac{1}{\mu} + \displaystyle\int_0^\pi f(w)\sin\theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}} - \epsilon}\,\mathrm{d}t. \end{cases} \tag{3.7}$$

However, by (1.24) and (1.25),

$$\lim_{n\to\infty}\frac{2}{\lambda_n}\int_{\alpha(\epsilon)}^\pi f_{\lambda_n}(t)\,\mathrm{d}t = \lim_{n\to\infty}\frac{2}{\lambda_n}\int_0^\pi f_{\lambda_n}(t)\,\mathrm{d}t = \lim_{n\to\infty}\frac{-2}{\lambda_n\Lambda_n}\int_0^\pi q'_{\lambda_n}(t)\,\mathrm{d}t = \lim_{n\to\infty}\Lambda_n^{-1} = \frac{\pi}{2h}. \tag{3.8}$$

Also, from (iii),

$$\lim_{n\to\infty}\left(\frac{1}{\mu_n} + \int_0^\pi f_{\lambda_n}(w)\sin\theta_n(w)\,dw\right)^{\frac{1}{3}} = \left(\frac{1}{\mu} + \int_0^\pi f(w)\sin\theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}. \tag{3.9}$$

Collecting (3.5)–(3.9), we find that

$$\frac{\pi\sqrt{(3g)}}{2h}\left(\frac{2h}{\pi}\right)^{\frac{3}{2}}\left[\left(\frac{1}{\mu} + \int_0^\pi f(w)\sin\theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}} - \epsilon\right]^{\frac{3}{2}} \leqslant \lim_{n\to\infty} c(\mu_n, \theta_n)$$

$$\leqslant \frac{\pi\sqrt{(3g)}}{2h}\left(\frac{2h}{\pi}\right)^{\frac{3}{2}}(1-\epsilon)^{-\frac{3}{2}}\left(\frac{1}{\mu} + \int_0^\pi f(w)\sin\theta(w)\,\mathrm{d}w\right)^{\frac{1}{2}}.$$

and since ϵ is arbitrary, it follows that

$$\lim_{n\to\infty} c(\mu_n, \theta_n) = \sqrt{\left\{\frac{6gh}{\pi}\left(\frac{1}{\mu} + \int_0^\pi f(w)\sin\theta(w)\,\mathrm{d}w\right)\right\}^{\frac{1}{2}}}.$$

That this last quantity lies in an interval $(\sqrt{(gh)}, 2\sqrt{(gh)}$ has been established in I (theorems 3.9, 4.12, and the footnote to theorem 3.7 (c)).

(v) An analogous calculation to that just given yields (v).

Corollary 3.4. *For each β, $0 < \beta < \frac{1}{6}\pi$, and $h, \lambda > 0$ there exists on a flow of mean depth h, a periodic, symmetric water-wave of wavelength λ, the free surface of which subtends an angle β with the horizontal at its steepest point. If h is fixed and $\lambda_n \uparrow \infty$ as $n \to \infty$, then a subsequence of the periodic wave profiles converge uniformly on compact subsets of $\mathbb{R}$ to the profile of a steady solitary water-wave whose free surface subtends a maximum angle of β with the horizontal, and whose asymptotic depth is h.*

Proof. By theorem 2.2, there exists $(\mu_n, \theta_n) \in \mathscr{C}_{\lambda_n}$ such that $|\theta|_{C_0[0,\pi]} = \beta$, for any $\beta \in [0, \pi/6)$. The result will follow by the method used in the proof of Theorem 3.1, once it is established that the sequence $\{\mu_n\}$ is bounded. However

$$\theta_n(s) = \frac{2}{3}\int_0^\pi G(s,t)\,\frac{f_{\lambda_n}(t)\sin\theta_n(t)}{\dfrac{1}{\mu_n}+\displaystyle\int_0^t f_{\lambda_n}(w)\sin\theta_n(w)\,\mathrm{d}w}\,\mathrm{d}t$$

$$\geqslant \frac{2}{3}\int_0^\pi G(s,t)\,\frac{f_{\lambda_m}(t)\sin\theta_n(t)}{\dfrac{1}{\mu_n}+\displaystyle\int_0^t f_{\lambda_m}(w)\sin\theta_n(w)\,\mathrm{d}w}\,\mathrm{d}t,$$

if $n \geqslant m$, by lemma 3.2. Without loss of generality suppose that $\mu_n \to \infty$. Then, as in the proof of theorem 2.2 (iv), it can be shown that there exists $B > 0$ such that $\theta_n(s) \geqslant B \sin s$, for all $s \in [0, \pi]$. But this estimate is enough to guarantee (by a routine adaptation of the methods of (I; § 5)) that a subsequence $\{\theta_{n(k)}\}$ of $\{\theta_n\}$ converges in $C[\delta, \pi]$, for each $\delta \in (0, \pi)$, to a non-trivial solution θ of the equation

$$\theta(s) = \frac{2}{3}\int_0^\pi G(s,t)\,\frac{f(t)\sin\theta(t)}{\displaystyle\int_0^t f(w)\sin\theta(w)\,\mathrm{d}w}\,\mathrm{d}t.$$

However, we know from I, theorem 5.2 that for such a function θ, $\limsup_{s\to 0^+}\theta(s) \geqslant \frac{1}{6}\pi$. This contradicts the fact that $|\theta_n|_{C_0[0,\pi]} = \beta < \frac{1}{6}\pi$.

Finally, we have the following result. Let $\mathscr{S} = \{(\mu, \theta) \in (0, \infty) \times \mathscr{K}_0\colon (\mu, \theta)$ solves (1.47) and $\theta \neq 0\}$. For all $(\mu, \theta) \in \mathscr{S}$, the product $f\theta \in L_1(0, \pi)$ (I, theorem 4.1). Let $\mathscr{S}' = \{(\mu, \theta) \in \mathscr{S}\colon (\mu, \theta)$ is the limit, as $\lambda \to \infty$, in $\mathbb{R} \times C_0[0, \pi]$ of a sequence $(\mu_\lambda, \theta_\lambda)$, where $(\mu_\lambda, \theta_\lambda) \in \mathscr{C}_\lambda\}$.

THEOREM 3.5. *If $\mathscr{C}'$ is the maximal connected subset of $\mathscr{S}'$ which contains* $(6/\pi, 0)$, *then $\mathscr{C}'$ is closed, unbounded, and has all the properties attributed to $\mathscr{C}$ in* I, *theorem* 3.9. *Clearly $\mathscr{C}' \subset \mathscr{C}$.*

Proof. This is immediate, since it has been shown that the boundary ∂U of every bounded, open set $U \subset \mathbb{R} \times C_0[0, \pi]$ which contains $(6/\pi, 0)$, contains a point of $\mathscr{C}'$. Since the set $\mathscr{S}'$ is obviously a closed subset of $\mathscr{S}$, and it has the property that bounded subsets of it are relatively compact (I, theorem 3.8), the result is immediate from I, theorem A 6.

Remarks. (*a*) Section 4 of I gives further properties of the elements of $\mathscr{C}$. In particular, the function θ is real-analytic on $[0, \pi)$, and so the wave profile is an analytic curve in $\mathbb{R}^2$, and the rate at which the free-surface approaches its asymptotic level is estimated. In § 5 of I, it is shown that if $\{(\mu_n, \theta_n)\} \subset \mathscr{C}'$ and $\mu_n \to \infty$ as $n \to \infty$, then a subsequence converges to a non-trivial 'solitary wave of extreme form' which satisfies (1.47) with $\mu = \infty$. The behaviour of this wave at its crest is similar to that given in theorem 2.2 (vi).

Clearly the results of McLeod (1982), quoted in theorem 2.2 (vii) for periodic waves, hold also for solitary waves corresponding to $\mathscr{C}'$, or $\mathscr{C}$. This agrees with numerical results (Longuet-Higgins & Fox 1977, p. 738).

(*b*) Since periodic waves converge to solitary waves on compact sets as the wavelength goes to infinity, it is reasonable to hope that the limiting solitary wave will inherit some of the properties of periodic waves given in § 2.3. Unfortunately, this has *not* been proved for the conclusions of theorem 2.5; only some parts of theorem 2.3 hold in the solitary wave case. The difficulty lies in the fact that $R_\lambda \to R_\infty$, while the uniform convergence of periodic waves to solitary waves is only on compact intervals. If, however, the plan outlined in the remark following theorem 2.6 could be implemented, then conclusions would follow which would be compatible with the numerical results on solitary waves (Byatt-Smith & Longuet-Higgins 1976, p. 185), on the convergence of periodic waves to solitary waves (Cokelet 1977), and on the solitary wave of extreme form (Longuet-Higgins 1974, p. 10).

The results of theorem 2.4 do go over in the limit as $\lambda \to \infty$, and one can show that elements of $\mathscr{C}'$ satisfy $\chi\Theta'(\chi) < \Theta(\chi)$ and $\Theta(\chi) < \frac{1}{3}\pi$ on $(0, \infty)$.

Appendix. On periodic flows of infinite depth

Theorem. *Suppose that θ is an odd, continuous function on $[-\pi, \pi]$ with $0 < \theta(s) < \pi$ on $(0, \pi)$, which satisfies the integral equation*

$$\theta(s) = \frac{1}{6}\int_{-\pi}^{\pi} \frac{1}{\pi}\ln\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right| \frac{\sin\theta(t)}{\dfrac{1}{\mu}+\displaystyle\int_0^t \sin\theta(w)\,\mathrm{d}w}\,\mathrm{d}t \tag{A 1}$$

for some $\mu > 0$. Then θ is real-analytic on $[-\pi, \pi]$ and $0 < \theta(s) < \frac{1}{3}\pi$ on $(0, \pi)$. Moreover, $\mu > 3$, and if λ and c are positive real numbers such that

$$\left(\frac{3g\lambda}{2\pi c^2}\right)^{\frac{1}{3}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos\theta(t)}{\left(\dfrac{1}{\mu}+\displaystyle\int_0^t \sin\theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}t, \tag{A 2}$$

then there exists a periodic wave of wavelength λ on a flow of infinite depth. The velocity of the flow at infinite depth is then c, and its speed at the crest is given by

$$q_c = (3g\lambda c/2\pi\mu)^{\frac{1}{3}}.$$

The free surface may be parametrized by $(x, H_\lambda(x))$, where $x \in (-\frac{1}{2}\lambda, \frac{1}{2}\lambda)$ and for $x \in [0, \frac{1}{2}\lambda]$

$$H_\lambda(x) - H_\lambda(0) = \left(\frac{\lambda^2c^2}{3g\pi^2}\right)^{\frac{1}{3}} \int_{\alpha^{-1}(x)}^{0} \frac{\frac{1}{2}\sin\theta(t)}{\left(\dfrac{1}{2\mu}+\displaystyle\int_0^t \tfrac{1}{2}\sin\theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}t, \tag{A 3}$$

where for $s \in [-\pi, 0]$

$$\alpha(s) = \left(\frac{\lambda^2c^2}{3g\pi^2}\right)^{\frac{1}{3}} \int_{s}^{0} \frac{\frac{1}{2}\cos\theta(t)}{\left(\dfrac{1}{2\mu}+\displaystyle\int_0^t \tfrac{1}{2}\sin\theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}t. \tag{A 4}$$

Proof. The proof that θ is real-analytic and bounded by $\frac{1}{3}\pi$ follows as in theorems 2.2 and 2.4. To show that $\mu > 3$, multiply (A 1) by $\sin s$ and integrate over $(-\pi, \pi)$, using (1.27).

As before, there exists a harmonic function $\tilde{\theta}$ on the unit disc such that $\tilde{\theta}(\mathrm{e}^{\mathrm{i}s}) = \theta(s)$ for all $s \in (-\pi, \pi]$, and

$$\left.\frac{\partial\tilde{\theta}}{\partial r}\right|_{\mathrm{e}^{\mathrm{i}s}} = \frac{1}{3}\frac{\sin\theta(s)}{\dfrac{1}{\mu}+\displaystyle\int_0^s \sin\theta(w)\,\mathrm{d}w}. \tag{A 5}$$

Using the expansion of G given in (1.27), it follows from (A 1) that for all $s \in (-\pi, \pi]$,

$$\theta(s) = \frac{1}{3}\int_{-\pi}^{\pi}\left\{\frac{1}{\pi}\sum_{l=1}^{\infty}\frac{\sin ls\,\sin lt}{l}\right\}\frac{\sin\theta(t)}{\dfrac{1}{\mu}+\displaystyle\int_0^t \sin\theta(w)\,\mathrm{d}w}\,\mathrm{d}t.$$

From this and Fubini's theorem there results that the Fourier series for θ is

$$\sum_{l=1}^{\infty} a_l \sin ls, \tag{A 6}$$

where
$$a_l = -\frac{1}{3\pi}\int_{-\pi}^{\pi} \cos lt \ln\left(\frac{1}{\mu}+\int_0^t \sin\theta(w)\,\mathrm{d}w\right)\mathrm{d}t. \tag{A 7}$$

It follows that putting
$$\tilde{\tau}(\xi)+\mathrm{i}\tilde{\theta}(\xi) = \sum_{l=1}^{\infty} a_l \xi^l \tag{A 8}$$

defines an analytic function on the unit disc, and

$$\tilde{\tau}(\mathrm{e}^{\mathrm{i}s})+\mathrm{i}\tilde{\theta}(\mathrm{e}^{\mathrm{i}s}) = a_0 - \tfrac{1}{3}\ln\left(\frac{1}{\mu}+\int_0^s \sin\theta(w)\,\mathrm{d}w\right)+\mathrm{i}\theta(s) \tag{A 9}$$

for all $s\in[-\pi,\pi]$, where $a_0 = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \tfrac{1}{3}\ln\left(\frac{1}{\mu}+\int_0^t \sin\theta(w)\,\mathrm{d}w\right)\mathrm{d}t.$

Let c and λ be positive real numbers chosen so that (A 2) holds. Then an analytic function $\tilde{T}-\mathrm{i}\tilde{\Theta}$ can be defined on $R_\lambda = \{\chi+\mathrm{i}\eta\colon -\frac{1}{2}\lambda<\chi<\frac{1}{2}\lambda, \eta<0\}$ by putting

$$\tilde{T}(\zeta)-\mathrm{i}\tilde{\Theta}(\zeta) = \tilde{\tau}(\exp(-2\pi\mathrm{i}\zeta/\lambda))+\mathrm{i}\tilde{\theta}(\exp(-2\pi\mathrm{i}\zeta/\lambda)).$$

Hence $\tilde{\Theta}(\chi+\mathrm{i}0) = -\theta(-2\pi\chi/\lambda)$ and so

$$\left.\frac{\partial\tilde{\Theta}}{\partial\eta}\right|_{\chi+\mathrm{i}0} = \frac{2\pi}{3\lambda}\,\frac{-\sin\theta(-2\pi\chi/\lambda)}{\dfrac{1}{\mu}+\displaystyle\int_0^{-2\pi\chi/\lambda}\sin\theta(w)\,\mathrm{d}w} = \frac{1}{3}\,\frac{\sin\Theta(\chi)}{\dfrac{\lambda}{2\pi\mu}+\displaystyle\int_0^{\chi}\sin\Theta(w)\,\mathrm{d}w}, \tag{A 10}$$

where $\Theta(x) = \tilde{\Theta}(x+\mathrm{i}0)$. Since $|\theta|<\frac{1}{3}\pi$ on $[-\pi,\pi]$, it follows by the maximum principle that $|\tilde{\Theta}|<\frac{1}{3}\pi$ in R_λ.

Now define an analytic function $\tilde{m}$ on R_λ by putting

$$\tilde{m}(\zeta) = \int_0^{\zeta}\exp(\tilde{T}(\zeta')-\mathrm{i}\tilde{\Theta}(\zeta'))\,\mathrm{d}\zeta'.$$

Since $\tilde{\Theta}(\pm\frac{1}{2}\lambda+\mathrm{i}\eta) = 0$ for all $\eta<0$, and since $|\tilde{T}(\zeta)-\mathrm{i}\tilde{\Theta}(\zeta)|\to 0$ as $|\zeta|\to\infty$, $\zeta\in R_\lambda$, it follows that $\tilde{m}$ is a conformal mapping from R_λ onto an infinite region in the z-plane of the form $S_\lambda = \{x+\mathrm{i}y\colon -\frac{1}{2}\lambda<x<\lambda, y<H_\lambda(x)\}$, and $H_\lambda'(x) = -\tan\Theta(\tilde{m}^{-1}(x+\mathrm{i}H_\lambda(x)))$. Since $\tilde{m}$ is invertible, we can define a complex potential $\tilde{\omega} = \tilde{\phi}+\mathrm{i}\tilde{\psi}$ on S_λ by putting

$$\tilde{\omega}(z) = c\tilde{m}^{-1}(z),$$

where c was chosen when λ was chosen so that (A 2) holds. Then for $z\in S_\lambda$,

$$\begin{aligned} u(z)-\mathrm{i}v(z) &= -\frac{\mathrm{d}\omega}{\mathrm{d}z} \\ &= -c\exp(-\tilde{T}(\tilde{m}^{-1}(z)))\{\cos\tilde{\Theta}(\tilde{m}^{-1}(z))+\mathrm{i}\sin\tilde{\Theta}(\tilde{m}^{-1}(z))\} \end{aligned}$$

and it follows that $c\exp(-\tilde{T}(\tilde{m}^{-1}(z)))$ is the speed of the flow and $-\Theta(\tilde{m}^{-1}(z))$ is the angle which the negative velocity vector makes with the x-axis at a point $z\in S_\lambda$. Moreover $u(z)-\mathrm{i}v(z)\to -c$ as $|z|\to\infty$, $z\in S_\lambda$. From the definition of $\tilde{\omega}$, it follows that $\tilde{\psi}\to-\infty$ as $|z|\to\infty$, $z\in S_\lambda$, and $\tilde{\psi} = 0$ on the free surface $\Gamma_\lambda = \{(x, H_\lambda(x))\colon x\in(-\frac{1}{2}\lambda,\frac{1}{2}\lambda)\}$.

Finally to show that the free surface condition is satisfied, we proceed as follows. By (A 8), (A 9) and Cauchy's formula there results that

$$1 = \exp(0) = \frac{1}{2\pi \mathrm{i}}\int_{-\pi}^{\pi}\frac{\exp(\tilde{\tau}(\mathrm{e}^{\mathrm{i}t})+\mathrm{i}\tilde{\theta}(\mathrm{e}^{\mathrm{i}t}))\,\mathrm{i}\mathrm{e}^{\mathrm{i}t}}{\mathrm{e}^{\mathrm{i}t}}\,\mathrm{d}t = \frac{\exp(a_0)}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\theta(t)}{\left(\dfrac{1}{\mu}+\displaystyle\int_0^t\sin\theta(w)\,\mathrm{d}w\right)^{\frac{1}{3}}}\,\mathrm{d}t,$$

and so, by our choice of λ and c,

$$\exp(a_0) = (2\pi c^2/3g\lambda)^{\frac{1}{3}}. \tag{A 11}$$

Hence

$$\tilde{\tau}(\mathrm{e}^{\mathrm{i}s}) = \tfrac{1}{3}\ln\left(\frac{2\pi c^2}{3g\lambda}\right)-\tfrac{1}{3}\ln\left(\frac{1}{\mu}+\int_0^s\sin\theta(w)\,\mathrm{d}w\right),$$

and so

$$\tilde{T}(\chi+\mathrm{i}0) = \tfrac{1}{3}\ln\left(\frac{2\pi c^2}{3g\lambda}\right)-\tfrac{1}{3}\ln\left(\frac{1}{\mu}+\frac{2\pi}{\lambda}\int_0^\chi\sin\Theta(w)\,\mathrm{d}w\right). \tag{A 12}$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{d}\chi}\left\{\frac{c^2}{2}\exp(-2\tilde{T}(\chi+\mathrm{i}0))+g\,\mathrm{Im}\,\tilde{m}(\chi+\mathrm{i}0)\right\}$$

$$= \frac{2\pi}{3\lambda}\,\frac{c^2\exp(-2\tilde{T}(\chi+\mathrm{i}0))\sin\Theta(\chi)}{\dfrac{1}{\mu}+\dfrac{2\pi}{\lambda}\displaystyle\int_0^\chi\sin\Theta(w)\,\mathrm{d}w}-g\exp(\tilde{T}(\chi+\mathrm{i}0))\sin\tilde{\Theta}(\chi+\mathrm{i}0)$$

$$= \exp(\tilde{T}(\chi+\mathrm{i}0))\left\{\frac{2\pi c^2}{3\lambda}\frac{3g\lambda}{2\pi c^2}\sin\Theta(\chi)-g\sin\Theta(\chi)\right\} = 0.$$

To complete the proof of the theorem we must verify that (A 3), (A 4) give the wave profile. This is a routine calculation based on the method used in the proof of theorem 1.5.

Results similar to those in theorems 2.4–2.6 hold if one replaces Θ by θ and $\frac{1}{2}\lambda$ by π.

Though the proof of this last theorem is in many respects similar to that of theorem 1.5, we have included it in order to obtain the following corollary. We need the notion of a conjugate function which is defined as follows. If u is an L_2-function whose Fourier series is $a_0+\sum_{l=1}^{\infty}(a_l\cos ls+b_l\sin ls)$, then the function conjugate to u is denoted by $\mathrm{C}u$ and is the L_2-function whose Fourier series is $\sum_{l=1}^{\infty}(a_l\sin ls-b_l\cos ls)$ (Bary 1964).

Corollary. *If θ satisfies* (A 1) *for some $\mu > 0$, then θ satisfies the equation*

$$\theta(s) = \tfrac{1}{6}\nu\int_{-\pi}^{\pi}\frac{1}{\pi}\ln\left|\frac{\sin\frac{1}{2}(s+t)}{\sin\frac{1}{2}(s-t)}\right|\exp(-3\mathrm{C}\theta(t))\sin\theta(t)\,\mathrm{d}t$$

where $\nu = 3g\lambda/2\pi c^2$.

Proof. By (A 6)–(A 9)

$$-\mathrm{C}\theta(t) = \tilde{\tau}(\mathrm{e}^{\mathrm{i}t}) = a_0-\tfrac{1}{3}\ln\left(\frac{1}{\mu}+\int_0^t\sin\theta(w)\,\mathrm{d}w\right),$$

whence by (A 11)

$$\exp(-3\mathrm{C}\theta(t)) = \frac{2\pi c^2}{3g\lambda}\left\{\frac{1}{\dfrac{1}{\mu}+\displaystyle\int_0^t\sin\theta(w)\,\mathrm{d}w}\right\}.$$

Substituting this last expression into (A 1) gives the required result.

Remark. In the previous sections, the mean depth was held fixed as $\lambda\to\infty$. If we now fix λ, and let $h\to\infty$, then one can prove a result analogous to theorem 3.1, but the proof is essentially simpler, because the limiting equation is non-singular. A word of caution is necessary however;

if $\{(\mu_n, \theta_n)\}$ is a sequence of solutions of (1.31) corresponding to waves of the same fixed wavelength λ, but of different mean depths $h_n \to \infty$, and if $\{(\mu_n, \theta_n)\} \subset \partial U$, where U is an open set in $\mathbb{R} \times C_0[0, \frac{1}{2}\lambda]$ containing (6, 0), then a subsequence converges to (μ, θ), where $(\frac{1}{2}\mu, \theta)$ is a solution of (A 1). This may be seen from (1.31), since (1.17) and (1.22) together give $f_\lambda(t) \to \frac{1}{2}$ uniformly for $t \in [-\pi, \pi]$, as $h \to \infty$.

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